On Mean Curvature Flow of Hypersurfaces

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Plan

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2. Mean Curvature Flow and its examples
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Levi-Civita connection

An **affine connection** on a smooth manifold M^n is a bilinear map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ such that for all $f \in C^\infty(M^n)$ and $X, Y \in \Gamma(TM)$,

1. $\nabla_{fX} Y = f \nabla_X Y$
2. $\nabla_X (fY) = X(f)Y + f\nabla_X Y$

A **Riemannian manifold** is a smooth manifold M^n equipped with a smooth inner product g_p on $T_p M^n$ for each $p \in M^n$. In local coordinates we will write

$$g = g_{ij} dx^i \otimes dx^j$$

where $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$.

For any Riemannian manifold (M^n, g) there exists a unique connection ∇, called the **Levi-Civita connection** which satisfies the following

1. $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
2. $\nabla_X Y - \nabla_Y X = [X, Y]$

Hypersurfaces and Second Fundamental Form

Let M^n be a smooth n-dimensional manifold with a smooth immersion $X : M^n \to \mathbb{R}^{n+1}$. We will denote the image $X(M^n)$ by $\mathcal{M}^n$ which is a hypersurface when X is an embedding. We can induce a metric and a connection(Levi-Civita) on M^n from the standard metric and connection on $\mathbb{R}^{n+1}$. If $\{x^i\}$ is a local coordinate system on M^n, then we can define it by

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle dX\left(\frac{\partial}{\partial x^i}\right), dX\left(\frac{\partial}{\partial x^j}\right)\right\rangle = \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j}\right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the standard metric on $\mathbb{R}^{n+1}$, and

$$\nabla_u v = (D_u \tilde{v})^T$$

where $\tilde{v}$ is an extension of v, $D_{(\cdot)}(\cdot)$ is the standard connection on $\mathbb{R}^{n+1}$ and $(W)^T$ denotes the tangential component, i.e. $(\cdot)^T : T\mathbb{R}^{n+1} \to T\mathcal{M}^n$

Hypersurfaces and Second Fundamental Form

Let ν denote a unit normal (i.e. $\nu \in T(\mathcal{M}^n)^{\perp}$ and $|\nu| = 1$). The second fundamental form is a symmetric 2-tensor on M^n defined by

$$A_\nu(u, v) = \langle D_u(\tilde{v}), \nu \rangle$$

where $u, v \in \Gamma(TM^n|_U)$ are local vector fields and $\tilde{v}$ is an extension of v to an open set of $\mathbb{R}^{n+1}$. Using this, we define Weingarten map $L : TM^n \to TM^n$ by

$$g(L(X), Y) = A_\nu(X, Y)$$

Hypersurfaces and Second Fundamental Form

As A_ν is symmetric, the Weingarten map L is self-adjoint, so it can be diagonalized and has real eigenvalues. The eigenvalues are denoted by $\kappa_1, \ldots, \kappa_n$ and are called principal curvatures. The trace of the map L is called the mean curvature and is denoted by H,

$$H = \sum_{i=1}^{n} \kappa_i$$

One can also prove that the Weingarten map is equal to

$$L(u) = D_u N$$

where N is some extension of ν satisfying $|N| \equiv 1$.

Hypersurfaces and Second Fundamental Form

In local coordinates $\{x^i\}$, we can write

$$A_\nu = h_{ij} dx^i \otimes dx^j \text{ and } L(\partial_i) = L_i^j \partial_j$$

then

$$\begin{aligned} g(L(\partial_i), \partial_k) &= A_\nu(\partial_i, \partial_k) = h_{ik} \\ g(L_i^j \partial_i, \partial_k) &= h_{ik} \\ g_{jk} L_i^j &= h_{ik} \\ L_i^j &= g^{jk} h_{ik} \end{aligned}$$

where $[g^{ij}] = [g_{ij}]^{-1}$. In particular, $H = g^{ij} h_{ij}$.

Mean Curvature flow

MCF

A one-parameter family of immersion $X : M^n \times I \to \mathbb{R}^{n+1}$ is said to evolve by **Mean Curvature Flow** (MCF) if

$$\frac{\partial}{\partial t}X(p,t) = \vec{H}(x,t) = -H(x,t)\nu(x,t) \quad \forall (p,t) \in M^n \times I$$

Notice that the Mean Curvature vector $\vec{H} = -H\nu$ is independent of the direction of normal ν.

Let $\Delta_{\mathcal{M}_t}$ denote the Laplacian of the induced metric, then $\Delta_{\mathcal{M}_t}X = (\Delta_{\mathcal{M}_t}X_1, \ldots, \Delta_{\mathcal{M}_t}X_{n+1}) = -H\nu$ so MCF can be considered as heat type equation,

$$\frac{\partial}{\partial t}X(p,t) = \Delta_{\mathcal{M}_t}X$$

MCF as the negative gradient flow

Let $\mathcal{M}_0$ be a hypersurface in $\mathbb{R}^{n+1}$ and consider a variation $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ with $X_0 = \mathcal{M}_0$. Considering volume as function of time, we get

$$\frac{d}{dt}\mathrm{Vol}(\mathcal{M}_t) = \int_{\mathcal{M}_t} \langle \partial_t X, H\nu \rangle$$

Using this, the gradient of the volume functional is

$$\nabla \mathrm{Vol} = H\nu$$

so the most efficient way to reduce the volume is to choose the variation so that

$$\partial_t X = -\nabla \mathrm{Vol} = -H\nu$$

which is MCF. In particular, we get the following equation for evolution of volume under MCF,

$$\frac{d}{dt}\mathrm{Vol}(\mathcal{M}_t) = -\int_{\mathcal{M}_t} H^2$$

Short time existence

Short time existence : Let $X_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed manifold. There exists an $\epsilon > 0$ and a smooth solution $X : M^n \times [0, \epsilon) \to \mathbb{R}^{n+1}$ to MCF, with $X(\cdot, 0) = X_0$. Moreover, the solution is unique.

Examples

1 Shrinking spheres : $\mathcal{M}_t = S^n_{r(t)}$ where $r(t) = \sqrt{R_0^2 - 2nt}$ and the solution exists for $T = \frac{R_0^2}{2n}$.

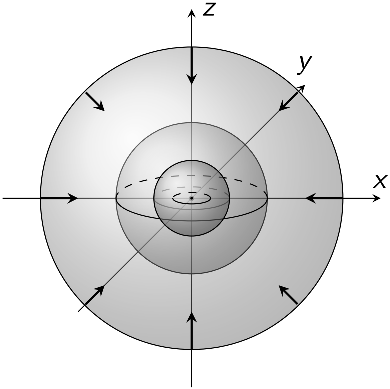

Figure: Shrinking spheres of dimension 2

2 Cylinder over solutions : If $\mathcal{M}_t^n \subset \mathbb{R}^{n+1}$ is a solution of MCF then so is $\mathcal{M}_t^n \times \mathbb{R}^m \subset \mathbb{R}^{n+m+1}$.

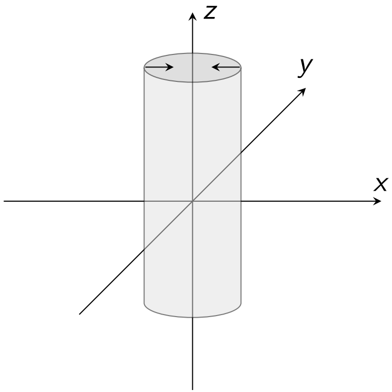

Figure: Cylinder $S^1 \times \mathbb{R}$

3 Minimal hypersurfaces : Any hypersurface with $H \equiv 0$ is a stationary solution of MCF.

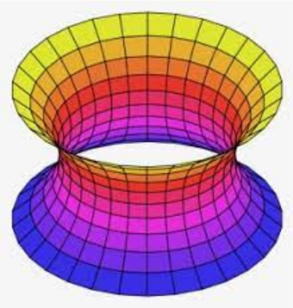

Figure: A catenoid

Evolution equations under MCF

In local coordinates we have the following evolution equations under MCF

$$\partial_t g_{ij} = -2Hh_{ij} \tag{0.1}$$

$$\partial_t h_{ij} = \Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij} \tag{0.2}$$

$$\partial_t H = \Delta H + |A|^2 H \tag{0.3}$$

$$\partial_t |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 \tag{0.4}$$

$$\begin{aligned}\partial_t \left(|A|^2 - \frac{1}{n}H^2\right) = \Delta \left(|A|^2 - \frac{1}{n}H^2\right) - 2\left(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\right) \\ + 2|A|^2 \left(|A|^2 - \frac{1}{n}H^2\right)\end{aligned} \tag{0.5}$$

In diagonalized frame, $|A|^2 = \left(\sum_{i=1}^n \kappa_i^2\right)$ and $H^2 = \left(\sum_{i=1}^n \kappa_i\right)^2$ so

$$|A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i<j}(\kappa_i - \kappa_j)^2$$

Maximum Principles

Scalar maximum principle

Let $g(t) \in [0, T)$ be a 1-parameter family of Riemannian metrics on a closed manifold $\mathcal{M}^n$ and $\beta : \mathcal{M}^n \times [0, T) \to \mathbb{R}$ be a locally bounded function. Let $u : \mathcal{M}^n \times [0, T) \to \mathbb{R}$ be a C^2 function satisfying the following inequality

$$\frac{\partial}{\partial t} u(x, t) \geq \Delta_{g(t)} u + \beta u$$

If $u(x, 0) \geq 0$ for all $x \in \mathcal{M}^n$, then $u(x, t) \geq 0$ for all $(x, t) \in \mathcal{M}^n \times [0, T)$.

Maximum principles

Using maximum principle we can prove that the compact manifolds with $H > 0$ must extinct in finite time.

Theorem

Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a maximal solution of the mean curvature flow with X_0 compact and with positive mean curvature everywhere. Then the mean curvature remains positive at all times and $T < \infty$.

Proof.

The mean curvature satisfies the equation

$$\partial_t H = \Delta H + |A|^2 H \geq \Delta H + \frac{1}{n} H^3$$

Let ψ be the solution of the ordinary differential equation

$$\frac{d\psi}{dt} = \frac{1}{n}\psi^3, \qquad \psi(0) = H_{min}(0) > 0$$

Proof.

Considering ψ as a function on $M^n \times [0, T)$ which is constant in the space slice, we get

$$\frac{\partial}{\partial t}\psi(H - \psi) \geq \Delta(H - \psi) + \frac{1}{n}(H^3 - \psi^3)$$

so by the maximum principle

$$H \geq \psi \qquad \text{for } t \in [0, T)$$

Solving for ψ gives

$$\psi(t) = \frac{H_{\min}(0)}{\sqrt{1 - (\frac{2}{n})H_{\min} t}}$$

which blows up in finite time.

□

Maximum Principles

Let $M = M_{ij} dx^i \otimes dx^j$ be a symmetric 2-tensor. We say M is non-negative if $v^T M v = M_{ij} v^i v^j \geq 0$ for all vectors v. Let $N_{ij} = p(M_{ij}, g_{ij})$ be a tensor formed by contracting products of M_{ij} with itself using the metric. Also suppose that whenever v is a null-eigenvector of M_{ij} (i.e. $M_{ij} v^j = 0$), we have $N_{ij} v^i v^j \geq 0$.

Tensor maximum principle

Let $g(t) \in [0, T)$ be a 1-parameter family of Riemannian metrics on a closed manifold $\mathcal{M}^n$. Let M_{ij} be a symmetric non-negative tensor evolving by the equation

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + N_{ij} \text{ for all } (x, t) \in \mathcal{M}^n \times [0, T)$$

where $N_{ij} = p(M_{ij}, g_{ij})$ satisfies the null-eigenvector condition above. If M is non-negative at $t = 0$, then it remains non-negative on $[0, T)$.

Maximum Principles

Corollary

Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ *be a compact solution of MCF such that* X_0 *is convex (*$A = h_{ij} dx^i \otimes dx^j$ *is non-negative). Then* X_t *is convex for all* $t \in [0, T)$.

Avoidance Principle

A geometric application of maximum principle is the avoidance principle. It says that if we start with two disjoint hypersurfaces and evolve them by MCF, then they remain disjoint for all time defined.
Let $X_1 : M_1^n \times [0, T) \to \mathbb{R}^{n+1}$ and $X_2 : M_2^n \times [0, T) \to \mathbb{R}^{n+1}$ be properly immersed solutions of MCF such that at least one of them is compact.
Define the distance function $d : M_1^n \times M_2^n \times [0, T) \to \mathbb{R}$ by

$$d(x, y, t) = |X_1(x, t) - X_2(y, t)|$$

Assume at time $t = 0$, the hypersurfaces are disjoint, so $X_1(M_1^n, 0) \cap X_2(M_2^n, 0) = \phi$, so $d_0 \doteqdot \inf_{(x,y)\in M_1^n \times M_2^n} d(x, y, 0) > 0$.

Avoidance Principle

Avoidance principle

Let X_1 and X_2 are solutions of mean curvature flow on closed manifolds. If $X_1(M_1^n, 0) \cap X_2(M_2^n, 0) = \phi$, which is equivalent to $d_0 > 0$ then

$$d(x, y, t) \geq d_0 \text{ for all } (x, y, t) \in M_1^n \times M_2^n \times [0, T)$$

Avoidance Principle

The avoidance principle gives another proof on the finite extinction time of compact hypersurface. Given any compact hypersurfaces, we can enclose it in a sphere of large radius without touching. Now the sphere collapses in finite time so by avoidance principle, the hypersurface must collapse in finite time too.

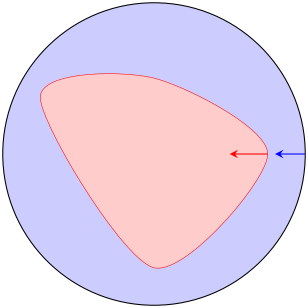

Figure: Compact hypersurface enclosed in a sphere

Huisken's theorem

Huisken's theorem

Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$, $n \geq 2$ be a maximal solution of MCF such that M^n is compact and $X_0 = X(\cdot, 0)$ is convex embedding. Then $X_t = X(\cdot, t)$ is a convex embedding for all $t > 0$ and X_t converges to a point $p \in \mathbb{R}^{n+1}$ as $t \to T$. Further the rescaled embeddings $\tilde{X}_t : M^n \to \mathbb{R}^{n+1}$ defined by

$$\tilde{X}_t(x) \doteqdot \frac{X_t(x) - p}{\sqrt{2n(T - t)}}$$

converge uniformly in the smooth topology to a smooth embedding whose image coincides with the unit sphere S^n.