Mean Curvature Flow

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Preface

This thesis is presented to Chennai Mathematical Institute in fulfillment of the thesis requirement for the degree Master of Science in Mathematics. The goal of this thesis is to give an introduction to mean curvature flow and explore some of its properties. The mean curvature flow is a PDE on a hypersurface immersed in Euclidean space. It is the negative of the area functional, so it flows hypersurface in the direction of their steepest descent of area functional. Similar to Ricci flow it's a heat-type equation, and we expect some uniformizing properties out of it. However, the flow develops singularity in finite time for mean-convex hypersurface. We look at some aspects of the singularity analysis which includes a monotonicity formula and convexity estimates.

Organization

The thesis is divided into three chapters where the first chapter serves as an introduction to the mean curvature flow. One of the crucial results done here is Huisken's monotonicity formula which describes the limit of type I singularities as a self-shrinker solution.

Following this, we study the asymptotic properties of the flow in the more general mean-convex setting in Chapter 2. Huisken-Sinestrari proved that asymptotically the flow converges to a weakly convex hypersurface.

Chapter 3 is on the Noncollapsing of mean-convex hypersurfaces. The result of Sheng-Wang and Andrews states that non-collapsing is preserved under mean curvature flow. The proof goes through deriving a differential inequality for inscribed curvature and using the maximal principle for viscosity solutions.

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Contents

1 Introduction to Mean curvature flow

We introduce the second fundamental form and mean curvature associated with an immersion of a hypersurface. The mean curvature flow is then the negative gradient flow of the volume functional on hypersurfaces.

1.1 Fundamentals of hypersurfaces

Let M^n be a smooth n-dimensional manifold with a smooth immersion $X : M^n \to \mathbb{R}^{n+1}$. If X is a homeomorphism onto its image, we say X is an **embedding** and its image $\mathcal{M}^n = X(M^n)$ has the structure of a smooth n-dimensional submanifold of $\mathbb{R}^{n+1}$. We say that M^n is an **immersed hypersurface** and $\mathcal{M}$ is an **embedded hypersurface** respectively. Throughout this thesis, we will denote the embedded manifold $X(M)$ by script $\mathcal{M}$ to differentiate between the domain and its image. Let $(U, \{x^i\})$ be a coordinate system on M^m, in Euclidean coordinates the pushforward of tangent vectors will be

$$dX(\partial_i) := \frac{\partial X}{\partial x^i} = \partial_i X$$

where $dX : TM^n \to T\mathbb{R}^{n+1}$ is the derivative of X. Since dX is an injection for each point in M^n, we can define an inner product on TM^n which in local coordinates is given by

$$g(\partial_i, \partial_j) = \langle \partial_i X, \partial_j X \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on Euclidean space. We will use the notation $\langle \cdot, \cdot \rangle$ for g as well which is consistent because of the immersion condition. Further, we can define the Levi-Civita connection on M^n from the Levi-Civita connection on $\mathbb{R}^{n+1}$. Let $X_p \in T_p\mathbb{R}^{n+1}$ be a vector and $Y : U \to T\mathbb{R}^{n+1}|_U$ be a local vector field in a neighborhood U containing p. The Levi-Civita connection of Y with respect to X on $\mathbb{R}^{n+1}$ is given by

$$D_{X_p} Y = (X_p(Y^1), \ldots, X_p(Y^{n+1}))$$

where $Y = (Y^1, \ldots, Y^{n+1})$ are the components of Y in the standard coordinates. Using the immersion condition, we define a connection on TM^n induced from D. Let $x \in M^n$ and $u \in T_x M^n, \tilde{v} \in TM^n|_U$ for some open set U containing x. Define a connection ∇ by

$$(\nabla_u \tilde{v}) = \pi(D_{dX(u)}(\tilde{V})) \tag{1.1.1}$$

where $\tilde{V}$ is an extension of $dX(\tilde{v})$ to an open set of $\mathbb{R}^{n+1}$ containing $X(U)$ and $\pi_{X(x)} : T_{X(x)}\mathbb{R}^{n+1} \to dX(T_x M)$ is the orthogonal projection onto the tangent subspace.

Lemma 1.1.1. The connection defined by Eq. (1.1.1) is well-defined and is the unique Levi-Civita connection on (M^n, g).

When X is an embedding, the restriction of the tangent bundle of $T\mathbb{R}^{n+1}|_{\mathcal{M}}$ can be decomposed as the direct sum

$$T\mathcal{M} \oplus N\mathcal{M}$$

where $N\mathcal{M}$ is the **normal bundle** which can be described as

$$N\mathcal{M} = \{(p, \nu) \in T\mathbb{R}^{n+1}|_{\mathcal{M}} : \langle u, \nu \rangle = 0 \text{ for all } u \in T_p\mathcal{M}\}.$$

For dimension reasons, the normal bundle at each point is one-dimensional. We fix a choice of unit normal ν_p for each $p \in \mathcal{M}$. This leads to **tangential projection** $\cdot^T : T\mathbb{R}^{n+1} \to T\mathcal{M}$ and **normal projection** $\cdot^\perp : T\mathbb{R}^{n+1} \to N\mathcal{M}$ maps of vectors in $T\mathbb{R}^{n+1}$ given by

$$u^T = u - \langle u, \nu \rangle \nu, \quad \text{and} \quad u^\perp = \langle u, v \rangle \nu$$

respectively. We can define the Levi-Civita connection on an embedded hypersurface $\mathcal{M}$ using the normal projection,

$$\nabla_u V = (D_u V)^T$$

where u is a vector and V is a local vector field. Notice that this is consistent with Eq. (1.1.1) since dX^{-1} is the tangential component. The next step is to calculate the Christoffel symbols of the connection ∇. For local coordinates $(U, \{x^i\})$ in M^n, the Christoffel symbols $\Gamma_{ij}^k : U \to \mathbb{R}$ are obtained by the formula

$$(\partial_i \partial_j X)^T = \Gamma_{ij}^k \partial_k X$$

Taking the inner product with $\partial_l X$ and inverting it we get

$$\Gamma_{ij}^k = g^{kl} \langle \partial_i \partial_j X, \partial_l X \rangle .$$

The normal part of the Euclidean covariant derivative is a tensor, called the **second fundamental form** of $\mathcal{M}$ and is denoted by A. Let $u, v \in T\mathcal{M}$ and $V \in \Gamma(T\mathcal{M})$ be an extension of v. Then A is given by

$$A(u, v) := (D_u V)^T$$

is independent of the extension V and is a symmetric two-tensor. The components of A over local coordinates will be denoted by $h_{ij} = A(\partial_i, \partial_j)$. Further we can use the isomorphism $T_p M \cong T_p^* M$ given by g to convert A into a linear map $L_p : T_p\mathcal{M} \to T_p\mathcal{M}$ given by

$$\langle L_p(u), v \rangle = A(u, v).$$

This is called the **Weingarten map**. Also if N is a local extension of the normal, then

$$L_p(u) = D_u N.$$

This is true because

$$0 = \frac{1}{2} D_u |N|^2 = \langle D_u N, \nu \rangle \quad \text{and} \quad \langle D_u N, v \rangle = -\langle \nu, D_u V \rangle = A(u, v)$$

where V is some extension of v. As A is symmetric, the Weingarten map is a self-adjoint operator with real eigenvalues. The (ordered) eigenvalues

$$\kappa_1(p) \leq \cdots \leq \kappa_n(p)$$

of L_p are called **principal curvatures**. If we switch the sign of the normal, the principal curvatures flip the sign as well since the Weingarten map flips the sign. The trace of the Weingarten map is called the **mean curvature** and is denoted by H. So in terms of principal curvatures,

$$H = \operatorname{tr}(L) = \kappa_1 + \cdots + \kappa_n.$$

We say that a hypersurface $X : M^n \to \mathbb{R}^{n+1}$ is **mean convex** if it admits a unit normal field with respect to which its mean curvature is non-negative and **strictly mean convex** if it admits a unit normal vector field with respect to which its mean curvature is positive.

We can express the Riemann curvature tensor in terms of principal curvatures as follows,

$$\begin{aligned} R_{ijkl} &= \left\langle \nabla^2_{ji} \partial_k - \nabla^2_{ij} \partial_k, \partial_l \right\rangle = h_{ik} h_{jl} - h_{il} h_{jk}, \\ Ric_{ij} &= g^{kl} R_{ikjl} = H h_{ij} - h_{il} g^{lk} h_{kj}, \\ R &= g^{ij} Ric_{ij} = g^{ij} g^{kl} R_{ikjl} = H^2 - |A|^2. \end{aligned}$$

Further, the principal curvatures satisfy some gradient formulas given as follows

Lemma 1.1.2. 1. (Codazzi's identity) $\nabla_i h_{jk} = \nabla_j h_{ik}$.

2. $\Delta h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij}$.

3. (Simon's identity) $\frac{1}{2} \Delta |A|^2 = \langle h_{ij}, \nabla_i \nabla_j H \rangle + |\nabla A|^2 + Z$ where $Z = H \operatorname{tr}(L^3) - |A|^4$.

1.2 Mean Curvature Flow

Now we define the mean curvature flow (MCF) on hypersurfaces.

Definition 1.2.1. A one-parameter family of immersion $X : M^n \times I \to \mathbb{R}^{n+1}$ is said to evolve by **mean curvature flow** (MCF) if

$$\frac{\partial}{\partial t}X(p,t) = \vec{H}(p,t) = -H(p,t)\nu(p,t) \quad \forall (p,t) \in M^n \times I. \tag{1.2.1}$$

Notice that the mean curvature vector $\vec{H} = -H\nu$ is independent of the direction of normal ν. The following lemma demonstrates the similarity of mean curvature flow with the heat equation

Lemma 1.2.1. The mean curvature vector is equal to the Laplace-Beltrami operator of the hypersurface

$$\vec{H} = -H\nu = \Delta_{\mathcal{M}} X.$$

Proof. Notice that $\partial_i\partial_j X = \Gamma_{ij}^k \partial_k X - h_{ij}\nu$. Contracting this,

$$\begin{aligned}\Delta_{\mathcal{M}} X &= g^{ij}\nabla_i\nabla_j X \\ &= g^{ij}(\partial_i\partial_j X - \Gamma_{ij}^k\partial_k X) \\ &= -g^{ij}h_{ij}\nu \\ &= -H\nu.\end{aligned}$$

□

We can generalize the definition to include only the normal part as well.

Definition 1.2.2. A one-parameter family of immersion $X : M^n \times I \to \mathbb{R}^{n+1}$ is said to evolve by **reparametrized mean curvature flow** (MCF) if

$$\left(\frac{\partial}{\partial t}X(p,t)\right)^{\perp} = \vec{H}(p,t) = -H(p,t)\nu(p,t) \quad \forall (p,t) \in M^n \times I. \tag{1.2.2}$$

The reason it is called so is that if we consider the one-parameter family of diffeomorphisms $\psi_t : M \to M$ generated by

$$\frac{\partial \psi_t}{\partial t}(p) = dX_t^{-1}\left(\left(\frac{\partial X}{\partial t}(\psi_t(p),t)\right)^{\perp} - \frac{\partial X}{\partial t}(\psi_t(p),t)\right), \qquad \psi_0 = \mathrm{id}_M$$

then the reparametrized manifold $\overline{X}_t = X_t \circ \psi_t$ is a solution of mean curvature flow.

The flow exists for a short time on any arbitrary hypersurface which is proved in [ACGL22]. This is known as the short-time existence of solutions

Theorem 1.2.2 (Short time existence)**.** Let $X_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of a compact manifold without boundary. There exists an $\epsilon > 0$ and a smooth solution $X : M^n \times [0,\epsilon) \to \mathbb{R}^{n+1}$ to MCF, with $X(\cdot,0) = X_0$. Moreover, the solution is unique.

1.2.1 Examples of the mean curvature flow

It is difficult to solve the mean curvature flow PDE on an arbitrary hypersurface. The limited number of examples come from ansatz or special cases,

1. **Shrinking spheres**: Let $\mathbb{S}^n(r) \subset \mathbb{R}^{n+1}$ be sphere of dimension n with radius r. Since the mean curvature $H = \frac{n}{r}$ is constant across the sphere, we make the ansatz that the hypersurface remains spherical under mean curvature flow. Let $\mathcal{M}_t = \mathbb{S}^n(r(t))$ be the solution, then the PDE is reduced to an ODE given by

$$\frac{d}{dt}r(t) = -\frac{n}{r(t)} \tag{1.2.3}$$

whose solution is $r(t) = \sqrt{r_0^2 - 2nt}$ with $r(0) = r_0$. So the shrinking spheres $\mathbb{S}^n(\sqrt{r_0^2 - 2nt})$ are a solution to the mean curvature flow for $t \in [0, \frac{r_0^2}{2n})$.

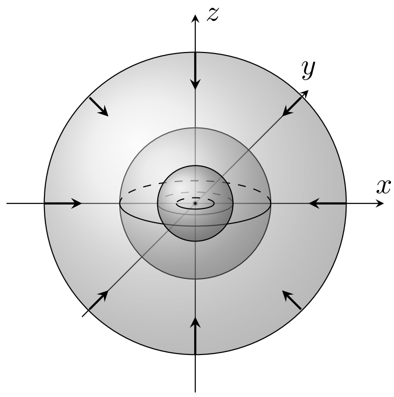

Figure 1.1: Shrinking spheres of dimension 2

2. **Evolution of Graphs**: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The graph of f in $\mathbb{R}^{n+1}$,

$$\mathcal{M} = \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}$$

is a smooth hypersurface. The mean curvature vector at $(x, f(x))$ for the hypersurface can be calculated by the formula

$$-H\nu + g^{ij}\Gamma_{ij}^k\partial_k X = g^{ij}\partial_i\partial_j X = \left(0, \sqrt{1+|\nabla f|^2}\text{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\right).$$

Ecker and Huisken proved in [EH89] that graphs evolving under the mean curvature flow remain graphs. So a family of graphs $\mathcal{M}_t = \{(x, f_t(x)) : x \in \mathbb{R}^{n+1}\}$ with the condition

$$\frac{\partial}{\partial t}f_t(x) = \sqrt{1+|\nabla f_t|^2}\,\text{div}\left(\frac{\nabla f_t}{\sqrt{1+|\nabla f_t|^2}}\right)$$

is a solution of the mean curvature flow (after a reparametrization).

3. **Minimal surfaces**: Minimal surfaces are the critical points of the volume functional. A hypersurface $\mathcal{M}$ is said to be a **minimal hypersurface** if it satisfies $H(x) = 0$ for all $x \in \mathcal{M}$. Hence, minimal hypersurfaces are stationary solutions of the mean curvature flow.

4. **Products of solutions with Euclidean space** : Suppose $\mathcal{M}_t^n \subset \mathbb{R}^{n+1}$ is a solution of the mean curvature flow. It is easy to verify that the mean curvature vector of the product $\mathcal{M}_t^n \times \mathbb{R}^m \subset \mathbb{R}^{n+1} \times \mathbb{R}^m$ is given by

$$\vec{H}(x,y) = (H(x)\nu(x), 0),$$

which implies that the time-parametrized product $\mathcal{N}_t = \mathcal{M}_t \times \mathbb{R}^{n+1}$ is a solution of the mean curvature flow as well.

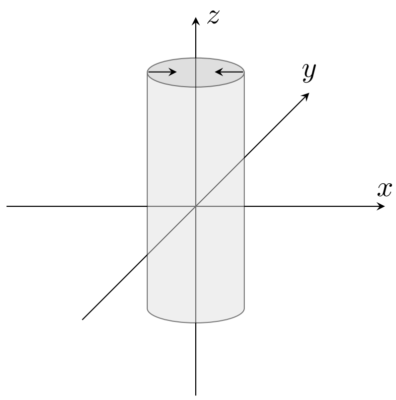

Figure 1.2: Cylinder $S^1 \times \mathbb{R}$

1.2.2 Mean curvature flow as the gradient of the area functional

Let $\mathcal{M}_0 \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ be a variation with $X(\cdot, 0) = \mathcal{M}_0$. Considering area as a function of time over the variation, we get

$$\frac{d}{dt}\text{Area}(\mathcal{M}_t) = \int_{\mathcal{M}_t} \langle \partial_t X, H\nu \rangle \tag{1.2.4}$$

Using this, the gradient of the area functional is

$$\nabla\text{Area} = H\nu$$

so the most efficient way to reduce the volume is to choose the variation so that

$$\partial_t X = -\nabla\text{Area} = -H\nu$$

which is the mean curvature flow. In particular, we get the following equation for the evolution of area under mean curvature flow,

$$\frac{d}{dt}\text{Area}(\mathcal{M}_t) = -\int_{\mathcal{M}_t} H^2$$

which is the steepest descent of area in the space of hypersurface up to speed-parametrization.

1.3 Evolution equations

To understand the properties of mean curvature flow it is essential to know the evolution of geometric quantities of the hypersurface. Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a smooth solution of mean curvature flow, so

$$\partial_t X(x,t) = \overrightarrow{H}(x,t) = -H(x,t)\nu(x,t).$$

The induced metric on the hypersurface is given by $g = X^*(\delta)$ where δ is the flat metric on $\mathbb{R}^{n+1}$. This means that if $\{x^i\}$ are local coordinates on M^n, then the components of the induced metric are given by

$$g_{ij} = \delta(X_*(\partial_i), X_*(\partial_j)) = \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle = \langle \partial_i X, \partial_j X \rangle .$$

Lemma 1.3.1. Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a solution of mean curvature flow. Then the evolution equation of the metric, normal, second fundamental form, and mean curvature is given by

$$\partial_t g_{ij} = -2Hh_{ij} \tag{1.3.1}$$

$$\partial_t \nu = \nabla H \tag{1.3.2}$$

$$\partial_t h_{ij} = \Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij} \tag{1.3.3}$$

$$\partial_t H = \Delta H + |A|^2 H \tag{1.3.4}$$

$$\partial_t |A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4 \tag{1.3.5}$$

Proof. 1. In local coordinates we have

$$\begin{aligned}
\partial_t g_{ij} &= \partial_t \langle \partial_i X, \partial_j X \rangle \\
&= \langle \partial_i(\partial_t X), \partial_j X \rangle + \langle \partial_i X, \partial_t(\partial_j X) \rangle \\
&= \langle \partial_i(-H\nu), \partial_j X \rangle + \langle \partial_i X, \partial_j(-H\nu) \rangle \\
&= -H \langle \partial_t \nu, \partial_j X \rangle - H \langle \partial_i X, \partial_t \nu \rangle \\
&= -2Hh_{ij}.
\end{aligned}$$

2. Since $\langle \nu, \nu \rangle = 1$, we have $2\langle \partial_t \nu, \nu \rangle = 0$, so the vector $\partial_t \nu$ is in the tangent plane of the hypersurface. We can write it as a linear combination of tangent vectors $\{\partial_j X\}$ to get

$$\begin{aligned}\partial_t \nu &= \langle \partial_t \nu, \partial_i X \rangle \partial_j X g^{ij} = - \langle \nu, \partial_i (\partial_t X) \rangle \partial_j X g^{ij} \\ &= \langle \nu, \partial_i (H\nu) \rangle \partial_j X g^{ij} \\ &= \partial_i H \partial_j X g^{ij} + H \langle \nu, \partial_i \nu \rangle \partial_j X g^{ij} \\ &= \partial_i H \partial_j X g^{ij} = \nabla H.\end{aligned}$$

3. From the relations

$$\partial_i \partial_j X = \Gamma^k_{ij} \partial_k X - h_{ij} \nu \quad \text{and} \quad \partial_j \nu = h_{jl} g^{lm} \partial_m X$$

we get

$$\begin{aligned}\partial_t h_{ij} &= -\partial_t \langle \partial_i \partial_j X, \nu \rangle \\ &= \langle \partial_i \partial_j (H\nu), \nu \rangle - \left\langle \partial_i \partial_j X, \partial_l H \partial_m X g^{lm} \right\rangle \\ &= \partial_i \partial_j H + H \left\langle \partial_i \left(h_{jm} g^{ml} \partial_l X \right), \nu \right\rangle - \left\langle \Gamma^k_{ij} \partial_k X - h_{ij} \nu, \partial_l H \partial_m X g^{lm} \right\rangle \\ &= \partial_i \partial_j H - \Gamma^k_{ij} \partial_k H + H h_{jm} g^{ml} \left\langle \Gamma^p_{il} \partial_p X - h_{il} \nu, \nu \right\rangle \\ &= \nabla_i \nabla_j H - H h_{il} g^{lm} h_{mj}.\end{aligned}$$

4. Utilizing the previous evolution equation with the product formula of derivatives,

$$\begin{aligned}\partial_t H &= \partial_t (g^{ij} h_{ij}) = (\partial_t g^{ij}) h_{ij} + g^{ij} \partial_t h_{ij} \\ &= -g^{ik} (\partial_t g_{kl}) g^{lj} h_{ij} + g^{ij} (\Delta h_{ij} - 2H h_{il} g^{lm} h_{mj} + |A|^2 h_{ij}) \\ &= -g^{ik} (-2H h_{kl}) g^{lj} h_{ij} + \Delta (g^{ij} h_{ij}) - 2H g^{ij} g^{lm} h_{il} h_{mj} + |A|^2 H \\ &= 2H|A|^2 + \Delta H - 2H|A|^2 + |A|^2 H \\ &= \Delta H + |A|^2 H.\end{aligned}$$

5. Again from the previous result on the evolution of h_{ij} we get

$$\begin{aligned}\partial_t |A|^2 &= \partial_t (g^{ik} g^{jl} h_{ij} h_{kl}) \\ &= 4H g^{im} g^{kn} h_{mn} g^{jl} h_{ij} h_{kl} + 2 g^{ik} g^{jl} h_{kl} (\Delta h_{ij} - 2H h_{im} g^{mn} h_{nj} + |A|^2 h_{ij}) \\ &= 2 g^{ik} g^{jl} h_{kl} \Delta h_{ij} + 2|A|^4,\end{aligned}$$

and

$$\Delta |A|^2 = g^{kl} \nabla_k \nabla_l (g^{pq} g^{mn} h_{pm} h_{qn}) = 2 g^{pq} g^{mn} h_{pm} \Delta h_{qn} + 2|\nabla A|^2.$$

□

Corollary. If mean curvature is positive everywhere on the initial hypersurface, then it remains so throughout the flow.

Proof. We apply the maximum principle to the evolution equation of H. □

Remark. This property of mean curvature holds even when the hypersurface is embedded in an arbitrary Riemannian manifold with positive Ricci curvature.

1.4 Maximum principle

We can extend the maximum principle on Euclidean space to general Riemannian manifolds in the following fashion. Refer to [CK04] for proofs.

Lemma 1.4.1 (Scalar maximum principle). Let $g(t) \in [0, T)$ be a 1-parameter family of Riemannian metrics on a closed manifold $\mathcal{M}^n$ and $\beta : \mathcal{M}^n \times [0, T) \to \mathbb{R}$ be a locally bounded function. Let $u : \mathcal{M}^n \times [0, T) \to \mathbb{R}$ be a C^2 function satisfying the following inequality

$$\frac{\partial}{\partial t} u(x, t) \geq \Delta_{g(t)} u + \beta u$$

If $u(x, 0) \geq 0$ for all $x \in \mathcal{M}^n$, then $u(x, t) \geq 0$ for all $(x, t) \in \mathcal{M}^n \times [0, T)$.

This can be generalized to include a non-linear term as well.

Lemma 1.4.2 (Comparison lemma). Let $u : M^n \times [0, T) \mapsto \mathbb{R}$ be C^2 function satisfying

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + \langle X, \nabla u \rangle + F(u)$$

on a closed manifold. Suppose there exists $C \in \mathbb{R}$ such that $u(x, 0) \geq C$ for all $x \in M^n$, and let ψ be the solution to the associated ordinary differential equation with initial condition

$$\frac{d\psi}{dt} = F(\psi), \qquad \psi(0) = C$$

Then $u(x, t) \geq \psi(t)$ for all $x \in M^n$ and $t \in [0, T)$ such that $\psi(t)$ exists.

The above-stated scalar maximum principle can be further extended to tensors. This was done by Hamilton in [Ham82] in the context of Ricci flow. Let $M = M_{ij} dx^i \otimes dx^j$ be a symmetric 2-tensor. We say M is non-negative if $v^T M v = M_{ij} v^i v^j \geq 0$ for all vectors v. Let $N_{ij} = p(M_{ij}, g_{ij})$ be a tensor formed by contracting products of M_{ij} with itself using the metric. Also suppose that whenever v is a null-eigenvector of M_{ij} (i.e. $M_{ij} v^j = 0$), we have $N_{ij} v^i v^j \geq 0$. Then the following maximum principle holds

Lemma 1.4.3 (Tensor maximum principle)**.** Let $g(t) \in [0,T)$ be a 1-parameter family of Riemannian metrics on a closed manifold $\mathcal{M}^n$. Let M_{ij} be a symmetric non-negative tensor evolving by the equation

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + u^k \nabla_k M_{ij} + N_{ij} \text{ for all } (x,t) \in \mathcal{M}^n \times [0,T)$$

where $N_{ij} = p(M_{ij}, g_{ij})$ satisfies the null-eigenvector condition above. If M is non-negative at $t = 0$, then it remains non-negative on $[0,T)$.

The tensor maximum principle can be used to prove that convexity is preserved under mean curvature flow.

Lemma 1.4.4. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a solution to mean curvature flow. If the initial hypersurface $X(\cdot,0) = \mathcal{M}_0$ is convex then $\mathcal{M}_t$ is also convex for all $t \in [0,T)$.

Proof. The condition of convexity is equivalent to $h_{ij} \geq 0$. We will prove a stronger result. Suppose that

$$\epsilon h g_{ij} \leq h_{ij}$$

for some constant $0 < \epsilon \leq \frac{1}{n}$ at $t = 0$, then the inequality remains true for $0 \leq t < T$. In the tensor maximum principle set

$$M_{ij} = \frac{h_{ij}}{H} - \epsilon g_{ij}, \qquad u^k = \frac{2}{H} g^{kl} \nabla_l H$$

and

$$N_{ij} = 2\epsilon H h_{ij} - 2h_{im} g^{ml} h_{lj}.$$

To calculate the time derivatives, we use Lemma 1.3.1,

$$\partial_t \left(\frac{h_{ij}}{H} \right) = \frac{H \Delta h_{ij} - h_{ij} \Delta H}{H^2} - 2h_{im} g^{ml} h_{ij},$$

$$\Delta \left(\frac{h_{ij}}{H} \right) = \frac{H \Delta h_{ij} - h_{ij} \Delta H}{H^2} - \frac{2}{H} g^{kl} \nabla_k \nabla_l \left(\frac{h_{ij}}{H} \right).$$

It remains to check that N_{ij} satisfies the null-eigenvector in this setting. Assume that for some vector $X = \{X^j\}$,

$$h_{ij} X^j = \epsilon H X_i$$

then

$$\begin{aligned} N_{ij} X^i X^j &= 2\epsilon H h_{ij} X^i X^j - 2h_{im} g^{ml} h_{lj} X^i X^j \\ &= 2\epsilon^2 H^2 |X|^2 - 2\epsilon^2 H^2 X^2 = 0. \end{aligned}$$

Substituting $\epsilon = 0$ yields the result. □

1.5 Long time existence

In this subsection, we will prove that the blow-up of the second fundamental form is the only obstruction to continuing the flow. The proof goes by contrapositive, relying on the Bernstein-type estimates. This technique is very similar to the one Hamilton used for Ricci flow [Ham82]. A solution of the mean curvature flow $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is said to be **maximal** if given any other solution $Y : M^n \times [0, S) \to \mathbb{R}^{n+1}$ which coincides with X for $t \in [0, T) \cap [0, S)$ we have $T \geq S$. Such a T is said to be the **maximal time** for X. This theorem characterizes the maximal time of existence.

Theorem 1.5.1. Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of the mean curvature flow with M^n compact. If X is maximal then $T < \infty$ and

$$\sup_{M^n \times [0,T)} |A| = \infty.$$

Before jumping into the proof we need a general notation for complicated tensor expressions occurring in evolution equations.

Definition 1.5.1. Given any two tensors A and B, we write $A*B$ to denote any linear combination of tensors formed by contraction on $A^{a_1 \dots a_p}_{i_1 \dots i_k} B^{b_1 \dots b_q}_{j_1 \dots j_l}$ with g or g^{-1}. The iterated product $A*B*C$ can be viewed as $A*(B*C)$ which is associative and can be written without brackets. Also, denote the multifold product $\underbrace{A * \cdots * A}_{p-\text{times}}$ by A^{*p}.
The Gauss equation in this notation yields $\mathrm{Rm} = A * A$ which after differentiation gives $\nabla \mathrm{Rm} = A * \nabla A$.

The following lemma will be necessary to find out the time derivative of covariant derivatives and gives the commutator relation between them.

Lemma 1.5.2. Let S be a tensor with an evolution equation given by

$$\partial_t S = \Delta S + T$$

where T is another tensor of the same rank. Then the evolution equation of the covariant derivative is

$$\partial_t \nabla S = \Delta \nabla S + A * A * \nabla S + A * \nabla A * S + \nabla T. \tag{1.5.1}$$

Proof. Recall the time evolution of Christoffel symbol is given by

$$\partial_t \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\nabla_i \partial_t g_{jl} + \nabla_j \partial_t g_{il} - \nabla_l \partial_t g_{ij}). \tag{1.5.2}$$

Substituting $\partial_t g_{ij} = -2Hh_{ij}$ we get

$$\partial_t \Gamma_{ij}^k = -g^{kl}(\nabla_i(Hg_{jl}) + \nabla_j(Hg_{il}) - \nabla_l(h_{ij})) = A * \nabla A.$$

Consider the commutator

$$\begin{aligned}\partial_t \nabla S &= \nabla \partial_t S + \partial_t \Gamma * S \\ &= \nabla(\Delta S + T) + A * \nabla A * S \\ &= \Delta \nabla S + \nabla \mathrm{Rm} * S + \mathrm{Rm} * \nabla S + \nabla T + A * \nabla A * S \\ &= \Delta \nabla S + A * A * \nabla S + A * \nabla A * S + \nabla T\end{aligned}$$

where we have used the Ricci identity $[\nabla, \Delta]S = \nabla \mathrm{Rm} * S + \mathrm{Rm} * \nabla A$. □

Lemma 1.5.3. The evolution equation of the higher gradient of the second fundamental form is given by

$$\partial_t \nabla^m A = \Delta \nabla^m A + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A \tag{1.5.3}$$

where $m \in \mathbb{N} \cup \{0\}$. Further, the norm of the gradient satisfies

$$\partial_t |\nabla^m A|^2 = \Delta |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A \tag{1.5.4}$$

where $m \in \mathbb{N} \cup \{0\}$.

Proof. We induct on m. For base case $m = 0$, the second fundamental form evolution equation is

$$\begin{aligned}\partial_t A &= \Delta A - 2HA^2 + |A|^2 A \\ &= \Delta A + A * A * A.\end{aligned}$$

Now suppose the equation holds for m, then for $m+1$ we have

$$\begin{aligned}\partial_t \nabla^{m+1} A &= \nabla \partial_t (\nabla^k A) + (\partial_t \Gamma) * \nabla^m A \\ &= \nabla \left(\Delta \nabla^m A + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A \right) + A * \nabla A * \nabla^m A \\ &= \Delta \nabla^{m+1} A + \nabla \mathrm{Rm} * \nabla^m A + \mathrm{Rm} * \nabla^{m+1} A + \sum_{i+j+k=m+1} \nabla^i A * \nabla^j A * \nabla^k A \\ &= \Delta \nabla^{m+1} A + \sum_{i+j+k=m+1} \nabla^i A * \nabla^j A * \nabla^k A\end{aligned}$$

using Ricci identity and Gauss equation $\mathrm{Rm} = A * A$. For the norm, we get

$$\partial_t |\nabla^m A|^2 = 2 \langle \partial_t \nabla^m A, \nabla^m A \rangle + A * A * \nabla^m A * \nabla^m A$$

where the second term comes from time derivative $\partial_t g^{ij} = -2Hh^{ij} = A * A$. This simplifies to

$$\begin{aligned}\partial_t |\nabla^m A|^2 &= 2 \left\langle \Delta \nabla^m A + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A, \nabla^m A \right\rangle + A * A * \nabla^m A * \nabla^m A \\ &= 2 \langle \Delta \nabla^m A, \nabla^m A \rangle + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A \nabla^m A \\ &= \Delta |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A\end{aligned}$$

□

We can consider the maximum principle on the previous lemma. This gives control of derivates of the second fundamental form based on the bound of the second fundamental form.

Lemma 1.5.4. Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of the mean curvature flow with M^n compact. Suppose that $T < \infty$ and $C_0 = \sup_{M^n \times [0,T)} |A| < \infty$. Then for each $m \in \mathbb{N}$ there exists a constant $C_m < \infty$ depending only on initial manifold such that

$$\sup_{M^n \times [0,T)} |\nabla^m A| \leq C_m. \tag{1.5.5}$$

Proof. From Eq. (1.5.4), we can estimate

$$\partial_t |\nabla^m A|^2 \leq \Delta |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + C_{n,m} \sum_{i+j+k=m} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A|$$

where $C_{n,m} < \infty$ only depends on n and m. We now proceed by induction on m. Suppose that for $l \in \{1, \dots, m-1\}$ we have

$$\sup_{\mathcal{M} \times [0,T)} |\nabla^l A|^2 \leq C_l$$

for some constants $C_l < \infty$. From the previous relation,

$$\begin{aligned}\partial_t |\nabla^m A|^2 &\leq \Delta |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + C_{n,m} \left(|A|^2 |\nabla^m A|^2 + \sum_{\substack{i+j+k=m \\ i,j,k \leq m-1}} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A| \right) \\ &\leq \Delta |\nabla^m A|^2 + C_{n,m} C_0^2 |\nabla^m A|^2 + c_m |\nabla^m A| \\ &\leq \Delta |\nabla^m A|^2 + 2C_{n,m} C_0^2 |\nabla^m A|^2 + \frac{c_m}{4C_0^2 C_{n,m}}\end{aligned} \tag{1.5.6}$$

using the induction hypothesis. To absorb the $|\nabla^m A|^2$ term on the right-hand side consider the inequality for $m-1$,

$$\begin{aligned}\partial_t|\nabla^{m-1}A|^2 &\leq \Delta|\nabla^{m-1}A|^2 - 2|\nabla^m A|^2 + C_{n,m-1}\sum_{i+j+k=m-1}|\nabla^i A||\nabla^j A||\nabla^k A||\nabla^{m-1}A| \\ &\leq \Delta|\nabla^{m-1}A|^2 - 2|\nabla^m A|^2 + c_{m-1}. \qquad (1.5.7)\end{aligned}$$

Multiplying Eq. (1.5.7) by $C_{n,m}C_0^2$ and adding to Eq. (1.5.6),

$$\begin{aligned}\partial_t\left(|\nabla^m A|^2 + C_{n,m}C_0^2|\nabla^{m-1}A|^2\right) \leq \Delta\left(|\nabla^m A|^2 + C_{n,m}C_0^2|\nabla^{m-1}A|^2\right) + \frac{c_m}{4C_0^2 C_{n,m}} \\ + c_{m-1}C_{n,m}C_0^2.\end{aligned}$$

Now by comparison principle, we get a bound

$$\sup_{\mathcal{M}^n\times[0,T)}\left(|\nabla^m A|^2 + C_{n,m}C_0^2|\nabla^{m-1}A|^2\right) \leq \overline{C_m}$$

and from this, we can deduce that

$$\sup_{\mathcal{M}\times[0,T)}|\nabla^m A|^2 \leq C_m$$

for some constant C_m depending only on $n, m, C_0, \ldots, C_{m-1}$ and the initial hypersurface. This completes the induction. □

We can improve the higher-order covariant derivative bound to include a time factor as well. The following lemma will imply a rapid decrease in the norm of higher covariant derivatives of the second fundamental form with respect to time.

Lemma 1.5.5. Let $X : M^n \times [0, r^2] \to \mathbb{R}^{n+1}$ be a solution of the mean curvature flow. Suppose there exists a constant $C_0 < \infty$ such that

$$|A|^2 \leq C_0 r^{-2}$$

on $\mathcal{M}^n \times [0, r^2]$. Then for each $m \in \mathbb{N}$, there exists a constant C_m depending only on $n, \mathcal{M}_0$ and C_0 such that

$$t^m|\nabla^m A|^2 \leq C_m r^{-2}$$

on $\mathcal{M}^n \times [0, r^2]$.

Proof. First, we will demonstrate how to obtain the bound for $m = 1$ and then extend the method for general m using induction. We know that

$$\begin{aligned}
\partial_t|\nabla A|^2 &= \Delta|\nabla A|^2 - 2|\nabla^2 A|^2 + \sum_{i+j+k=1} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A \\
&\leq \Delta|\nabla A|^2 - 2|\nabla^2 A|^2 + C_{n,1} \sum_{i+j+k=1} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla A| \\
&\leq \Delta|\nabla A|^2 - 2|\nabla^2 A|^2 + 3C_{n,1}|A|^2||\nabla A|^2 \\
&\leq \Delta|\nabla A|^2 - 2|\nabla^2 A|^2 + 3C_{n,1}C_0 r^{-2}|\nabla A|^2.
\end{aligned}$$

To obtain a better bound we want to utilize the good term $-2|\nabla^2 A|^2$ and one way to do this is to define another term with a t-factor and $|A|^2$,

$$F = t|\nabla A|^2 + \beta|A|^2$$

which is bounded at $t = 0$ by $\beta C_0 r^{-2}$. Notice that

$$\begin{aligned}
\partial_t F &\leq |\nabla A|^2 + t\left(\Delta|\nabla A|^2 - 2|\nabla^2 A|^2 + 3C_{n,1}C_0 r^{-2}|\nabla A|^2\right) + \beta\left(\Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4\right) \\
&\leq \Delta F - 2|\nabla^2 A|^2 + \left(1 + 3tC_{n,1}C_0 r^{-2} - 2\beta\right)|\nabla A|^2 + 2\beta C_0^2 r^{-4} \\
&\leq \Delta F + \left(1 + 3C_{n,1}C_0^2 - 2\beta\right)|\nabla A|^2 + 2\beta C_0^2 r^{-4}.
\end{aligned}$$

Choose $\beta > (1+3C_{n,1}C_0^2)/2$, so that the coefficient of $|\nabla A|^2$ is negative. The comparison theorem then gives

$$\sup_{x\in\mathcal{M}} F(x,t) \leq \beta C_0 r^{-2} + 2\beta C_0^2 r^{-4} t \leq C_1 r^{-2}$$

for some constant $C_1 > 0$. Hence,

$$t|\nabla A|^2 \leq C_1 r^{-2}$$

on $\mathcal{M} \times [0, r^2]$ which establishes the inequality for $m = 1$.

Now assume the inequality holds for $1, \ldots, m-1$. Then,

$$\begin{aligned}
\partial_t|\nabla^m A|^2 &\leq \Delta|\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + C_{n,m} \sum_{i+j+k=m} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A| \\
&\leq \Delta|\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + C_{n,m}\left(3|A|^2|\nabla^m A|^2 + \sum_{\substack{i+j+k=m \\ i,j,k\leq m-1}} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A|\right) \\
&\leq \Delta|\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + 3C_{n,m}C_0 r^{-2}|\nabla^m A|^2 + C_{n,m}L_m r^{-3} t^{-\frac{m}{2}}|\nabla^m A| \\
&\leq \Delta|\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + +C_{n,m} r^{-2}\left(6C_0|\nabla^m A|^2 + \frac{L_m^2 r^{-2} t^{-m}}{4C_0}\right) \\
&\leq \Delta|\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + r^{-2}(T_m|\nabla^m A|^2 + S_m r^{-2} t^{-m}) \\
&\leq \Delta|\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + U_m r^{-2}(|\nabla^m A|^2 + r^{-2} t^{-m})
\end{aligned}$$

where $L_m = \sum_{\substack{i+j+k=m \\ i,j,k\leq m-1}} \sqrt{C_iC_jC_k}$, $T_m = 6C_{n,m}C_0$, $S_m = (C_{n,m}L_m^2)/4C_0$ and $U_m = \max\{T_m, S_m\}$.

Also, from the induction hypothesis, there exist constants U_{m-k} for $1 \leq k \leq m$ such that

$$\partial_t|\nabla^{m-k}A|^2 \leq \Delta|\nabla^{m-k}A|^2 - 2|\nabla^{m-k+1}A|^2 + U_{m-k}r^{-4}t^{-(m-k)}$$

Now like in the $m = 1$ case, we define

$$F = t^m|\nabla^m A|^2 + \beta_m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} t^{m-k}|\nabla^{m-k}A|^2$$

where β_m is a positive constant to be determined later. Notice that at $t = 0$ we have a bound on F given by

$$F(x,0) \leq \beta_m(m-1)!C_0r^{-2}$$

Using the previously established estimates, the differential of F satisfies

$$\begin{aligned}\partial_t F \leq &\Delta F + U_m r^{-2}t^m|\nabla^m A|^2 + U_m r^{-4} + mt^{m-1}|\nabla^m A|^2 \\ &+ \beta_m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!}\left\{-2t^{m-k}|\nabla^{m-k+1}A|^2 + U_{m-k}r^{-4} + (m-k)t^{m-k-1}|\nabla^{m-k}A|^2\right\}.\end{aligned}$$

This expression is the reason we considered sharper estimate because the good terms

$$-2\frac{(m-1)!}{(m-k)!}t^{m-k}|\nabla^{m-k+1}A|^2$$

can be utilized to compensate for the bad terms

$$\frac{(m-1)!}{(m-k+1)!}(m-k+1)t^{m-k}|\nabla^{m-k+1}A|^2.$$

Collecting the rest of the terms yields

$$\partial_t F \leq \Delta F + (U_m r^{-2}t + m - 2\beta_m)t^{m-1}|\nabla^m A|^2 + (U_m + \beta_m V_m)r^{-4}$$

with $V_m = \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} U_{m-k}$. Choosing sufficiently large β_m, we can make the coefficient of $|\nabla^m A|^2$ negative which implies

$$\partial_t F \leq \Delta G + (U_m + \beta_m V_m)r^{-4}.$$

Now we are in the domain of the comparison principle which yields

$$\sup_{x\in\mathcal{M}} F(x,t) \leq \sup_{x\in\mathcal{M}} F(x,0) + (U_m+\beta_m V_m)r^{-4}t \leq (\beta_m C_0(m-1)! + U_m + \beta_m V_m)r^{-2} = C_m r^{-2}.$$

This implies the desired bound on $t^m|\nabla^m A|^2$ as

$$t^m|\nabla^m A|^2 \leq F \leq C_m r^{-2}$$

for $t \in [0, r^2]$. □

Remark. The parabolic nature of the mean curvature flow shows up here as the dimension of time is double the spatial dimension coming from $|\nabla^m A|$.

We finish up the proof for the stated long-time existence result.

Proof of Theorem 1.5.1. (Sketch) Assume on the contrary that $\sup_{M^n\times[0,T)}|A|^2 \le C$. We aim to show that the manifold $X(\cdot,t)=\mathcal{M}_t$ converges to a smooth limit $\mathcal{M}_T$ as $t\to T$. Notice that

$$|X(x,t)-X(x,0)| \le \left|\int_0^t H(x,s)ds\right| \le T\sup|H| \le T\sqrt{nC_0}$$

Moreover, notice that the time derivative of the metric satisfies

$$|\partial_t \log g(u,u)| = \left|2H\frac{A(u,u)}{g(u,u)}\right| \le 2nC^2$$

for any $u\in T_xM^n$. Integrating this we get

$$e^{-2nC^2T} \le \frac{g_{(x,t)}(u,u)}{g_{(x,0)}(u,u)} \le e^{2nC^2T}$$

for all $(x,t)\in\mathcal{M}\times[0,T)$. This proves that metrics at all times are uniformly equivalent. In fact, integrating from $T-\epsilon$ to T will yield that g_t converge to a continuous metric g_T which is also uniformly equivalent to metrics at previous times. Notice that X_t are diffeomorphic by the time translation map of the smooth solution of the mean curvature flow. Now the higher gradients of X can also be bounded from the bounds of higher derivatives of the second fundamental form. An Arzela-Ascoli argument now gives the smooth convergence $X_t\to X_T$. Applying short-time existence result at X_T we get a contradiction on the maximal time.

1.6 Monotonicity Formula

Mean curvature flow is invariant under parabolic scaling, i.e. if $X: M^n\times I\to\mathbb{R}^{n+1}$ is solution, then so is $X_\lambda(x,t)=\lambda X(x,\lambda^{-2}t)$. We construct a weighted area functional which is invariant under *parabolic* scaling along any solution to mean curvature flow which will be monotonous.

Let $\rho(x,t)$ be the backward heat kernel at (X_0,t_0), i.e.,

$$\rho(x,t)=\frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}}\cdot\exp\left(-\frac{|X(x,t)-X_0|^2}{4(t_0-t)}\right),\qquad t<t_0$$

Theorem 1.6.1 (Monotonicity formula). If M_t is a solution of mean curvature flow for $t < t_0$, then we have the formula

$$\frac{d}{dt}\int_{M_t} \rho(x,t)d\mu_t = -\int_{M_t} \rho(x,t)\left(H - \frac{\langle X(x,t) - X_0, \nu\rangle}{2(t_0 - t)}\right)^2 d\mu_t.$$

Proof. To simplify the formula assume that $(X_0, t_0) = (0, 0)$. We know that $\frac{d}{dt}\mu_t = -H^2\mu_t$, so differentiating ρ with respect to time we get,

$$\begin{aligned}
\frac{d}{dt}\int_{M_t} \rho(x,t)d\mu_t &= \int_{M_t} \rho(x,t)(-H^2)d\mu_t + \int_{M_t} \frac{\partial}{\partial t}\rho(x,t)d\mu_t \\
&= -\int_{M_t} \rho(x,t)H^2 d\mu_t + \int_{M_t} \left(\frac{\langle X(x,t), H(x,t)\nu\rangle}{2(-t)}\rho(x,t)\right) d\mu_t \\
&\quad + \int_{M_t} \left(\frac{n}{2(4\pi)(-t)}(4\pi)\rho(x,t) - \frac{|X(x,t)|^2}{4(-t)^2}\rho(x,t)\right) d\mu_t \\
&= \int_{M_t} \rho\left(\frac{n}{2(-t)} + \frac{\langle X, H\nu\rangle}{2(-t)} - \frac{|X|^2}{4(-t)^2} - H^2\right) d\mu_t. \qquad (1.6.1)
\end{aligned}$$

Now $\Delta X = -H\nu$, using this relation for the second term and divergence theorem we get

$$\begin{aligned}
\int_{M_t} \rho \langle X, H\nu\rangle d\mu_t &= -\int_{M_t} \rho \langle X, \Delta X\rangle d\mu_t \\
&= -\sum_{k=1}^{n+1}\int_{M_t} \rho X_k \Delta X_k d\mu_t \\
&= \sum_{k=1}^{n+1}\int_{M_t} \langle \nabla(\rho X_k), \nabla X_k\rangle d\mu_t \\
&= \sum_{k=1}^{n+1}\int_{M_t} (\langle \nabla\rho, \nabla X_k\rangle X_k + \rho\langle \nabla X_k, \nabla X_k\rangle) d\mu_t. \qquad (1.6.2)
\end{aligned}$$

Let $(U, \{x^i\})$ be some local coordinates on the hypersurface. In these coordinates we can write $\nabla\rho = g^{ij}\partial_i\rho\partial_j$, so $\langle \nabla\rho, \nabla X_k\rangle = \nabla\rho(X_k) = g^{ij}(\partial_i\rho)(\partial_j X_k)$ which implies

$$\begin{aligned}
\sum_{k=1}^{n+1} \langle \nabla\rho, \nabla X_k\rangle X_k &= \sum_{k=1}^{n+1} g^{ij}(\partial_i\rho)(\partial_j X_k)X_k \\
&= g^{ij}(\partial_i\rho)\langle X, \partial_j X\rangle \\
&= g^{ij}\rho\left(\frac{-\langle X, \partial_i X\rangle}{2(-t)}\right)\langle X, \partial_j X\rangle \\
&= -\frac{\rho}{2(-t)}|X^T|^2 \qquad (1.6.3)
\end{aligned}$$

and

$$\sum_{k=1}^{n+1} \rho \langle \nabla X_k, \nabla X_k \rangle = \sum_{k=1}^{n+1} \rho g^{ij} (\partial_i X_k)(\partial_j X_k) = \rho g^{ij} \langle \partial_i X, \partial_j X \rangle = \rho g^{ij} g_{ij} = n\rho. \quad (1.6.4)$$

Substituting Eq. (1.6.3) and Eq. (1.6.4) into Eq. (1.6.2) and multiplying by $\frac{1}{2(-t)}$, we get

$$\int_{M_t} \rho \frac{\langle X, H\nu \rangle}{2(-t)} d\mu_t = \int_{M_t} \rho \left(\frac{n}{2(-t)} - \frac{1}{4(-t)^2} |X^T|^2 \right) d\mu_t$$

or

$$\int_{M_t} \frac{n\rho}{2(-t)} d\mu_t = \int_{M_t} \rho \left(\frac{\langle X, H\nu \rangle}{2(-t)} + \frac{1}{4(-t)^2} |X^T|^2 \right) d\mu_t \quad (1.6.5)$$

where X^T denotes the tangential part of the vector X. Substituting Eq. (1.6.5) into Eq. (1.6.1)

$$\begin{aligned} \frac{d}{dt} \int_{M_t} \rho(x,t) d\mu_t &= \int_{M_t} \rho \left(\frac{\langle X, H\nu \rangle}{(-t)} - \frac{|X|^2}{4(-t)^2} - H^2 + \frac{1}{4(-t)^2} |X^T|^2 \right) d\mu_t \\ &= -\int_{M_t} \rho \left(H - \frac{\langle X, \nu \rangle}{2(-t)} \right)^2 d\mu_t. \end{aligned}$$

□

1.6.1 Rescaled Monotonicity formula

Applying the comparison principle on the time evolution of $|A|^2$ we get that if the curvature blows up at the maximal time T and satisfies the inequality

$$\max_{p \in M} |A(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}.$$

Using this we define the type I singularity if the blow-up rate is bounded by a constant.

Definition 1.6.1. Let T be the maximal time of existence of a mean curvature flow. If there exists a constant $C > 1$ such that

$$\max_{p \in M} |A(p,t)| \leq \frac{C}{\sqrt{2(T-t)}}$$

we say the flow is developing at time T a **type I singularity**. Conversely, if such a constant does not exist, that is

$$\limsup_{t \to T} \max_{p \in M} |A(p,t)| \sqrt{T-t} = \infty$$

we say that we have a **type II singularity**.

We will restrict ourselves to type I singularity for the rest of this section. We want to rescale M_t near a singular point as $t \to T$, such that the curvature of the rescaled surfaces remains uniformly bounded. Using the inequality $|H| \leq \sqrt{n}|A|$,

$$|X(p,t) - X(p,s)| \leq \int_s^t |H(p,\tau)| d\tau \leq \sqrt{n} C \left[(T-s)^{1/2} - (T-t)^{1/2} \right]$$

for all $p \in M^n$ and $0 \leq s < t < T$. Thus, $X(\cdot, t)$ converges uniformly as $t \to T$. Using this we define a blow-up point as follows

Definition 1.6.2. We say that $x \in \mathbb{R}^{n+1}$ is a **blow-up point** if there is $p \in M^n$ such that $X(p,t) \to x$ as $t \to T$ and $|A|(p,t)$ becomes unbounded as $t \to T$.

Let us assume now that $0 \in \mathbb{R}^{n+1}$ is a blow-up point. Then we define the rescaled immersions $\tilde{X}(p,s)$ by

$$\tilde{X}(p,s) = (2(T-t))^{-1/2} X(p,t), \quad s(t) = -\frac{1}{2}\log(T-t). \tag{1.6.6}$$

The surfaces $\tilde{M}_s = \tilde{X}(\cdot, s)(M^n)$ are therefore defined for $-\frac{1}{2}\log T \leq s < \infty$ and satisfy the equation

$$\frac{d}{ds}\tilde{X}(p,s) = \tilde{H}(p,s)\tilde{\nu}(p,s) + \tilde{X}(p,s).$$

For any tensor P, let $\tilde{P}$ denote the rescaled tensor. We say that P has degree α if $\tilde{P} = (2(T-t))^{-\frac{\alpha}{2}} P$. Then there is an expression evaluating the evolution of rescaled tensor.

Lemma 1.6.2. Suppose P is a tensor that satisfies

$$\frac{dP}{dt} = \Delta P + Q$$

for the original evolution equation and P has degree α. Then Q has degree $(\alpha - 2)$ and

$$\frac{d\tilde{P}}{ds} = \tilde{\Delta}\tilde{P} + \tilde{Q} + \alpha\tilde{h}\tilde{P}$$

For finding out the evolution of the second fundamental form and mean curvature after rescaling, we use this lemma to get.

Proposition 1.6.3. For each $m \geq 0$ there is a $C(m) < \infty$ such that $|\tilde{\nabla}^m \tilde{A}|^2 \leq C(m)$ holds on $\tilde{M}_s$ uniformly in s, where $C(m)$ depends on n, m, C_0 and M_0.

Proof. One can calculate the degree of $|\nabla^m A|^2 = -2(m+1)$ from Eq. (1.5.4) and further we know that

$$\partial_s |\tilde{\nabla}^m \tilde{A}|^2 \leq \Delta |\tilde{\nabla}^m \tilde{A}|^2 - 2|\tilde{\nabla}^{m+1}\tilde{A}|^2 + \tilde{C}_{n,m} \sum_{i+j+k=m} |\tilde{\nabla}^i \tilde{A}||\tilde{\nabla}^j \tilde{A}||\tilde{\nabla}^k \tilde{A}||\tilde{\nabla}^m \tilde{A}|$$

We induct on m. For $m = 0$, the rescaled second fundamental form under the type I condition satisfies

$$|\tilde{A}|^2 = 2(T-t)|A| \leq C$$

Assume the result holds for $0, \ldots, m-1$. Then for m there exists a constant B such that

$$\frac{\partial}{\partial s}|\tilde{\nabla}^m \tilde{A}|^2 \leq \tilde{\Delta}|\tilde{\nabla}^m A|^2 + B(1 + |\tilde{\nabla} A|^2)$$

Adding the Eq. (1.5.4) for $m-1$,

$$\frac{\partial}{\partial s}(|\tilde{\nabla}^m \tilde{A}|^2 + B|\tilde{\nabla}^{m-1}\tilde{A}|^2) \leq \tilde{\Delta}(|\tilde{\nabla}^m \tilde{A}|^2 + B|\tilde{\nabla}^{m-1}\tilde{A}|^2) - B|\tilde{\nabla}^m \tilde{A}|^2 + B_1$$

where B_1 depends on B and $\tilde{C}_{n,l}$ for $l = 0, \ldots, m-1$. Now from the induction hypothesis $|\tilde{\nabla}\tilde{A}|^2$ is bounded so from maximum principles in the previous estimate we can bound $|\tilde{\nabla}\tilde{A}|^2$ by a constant depending on initial data, B and B_1. This completes the induction. □

After the same rescaling the backward heat kernel becomes

$$\tilde{\rho}(p,s) = \exp\left(-\frac{1}{2}|\tilde{X}(p,s)|^2\right)$$

which leads to the following corollary.

Corollary. Let $\tilde{X}_s$ denote the rescaled hypersurfaces by the Eq. (1.6.6), then the corresponding monotonicity formula is

$$\frac{d}{ds}\int_{\tilde{X}_s} \tilde{\rho}\, d\tilde{\mu}_s = -\int_{\tilde{X}_s} \tilde{\rho}\left(\tilde{H} - \left\langle \tilde{X}, \tilde{\nu}\right\rangle\right)^2 d\tilde{\mu}_s.$$

Proof. We use the fact that $\frac{d\tilde{\mu}}{ds} = (n - H^2)\tilde{\mu}_s$ and calculate the derivative analogous to the proof in Theorem 1.6.1. □

One application of the estimates of the rescaled second fundamental form and rescaled monotonicity formula is to obtain a structural formula of type I singularity. This was proved by Huisken in [Hui90].

Theorem 1.6.4. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a maximal solution of mean curvature flow with type I singularity. Let $\tilde{X}_s$ denote the rescaled hypersurfaces with the origin as the blow-up point. Then for each sequence $s_j \to \infty$ there exists a subsequence s_{j_k} such that $\tilde{X}_{s_{j_k}}$ converges smoothly to an immersed non-empty limiting surface $\tilde{X}_\infty$ which satisfies

$$H = \left\langle \tilde{X}_\infty, \nu \right\rangle .$$

Further, if $\tilde{X}_\infty$ is compact, then is a sphere of radius $\sqrt{n}$.

2 Convexity estimates

As observed in the previous chapter the mean curvature flow preserves convexity and mean convexity. In this chapter, we would like to study the convexity of the hypersurface as it approaches singularity via a blow-up method. Huisken and Sinestrari proved in [HS99a, HS99b] that mean convex hypersurface are asymptotically convex i.e. blowing the flow near singularity gives a convex ancient solution.

2.1 Elementary symmetric polynomials and cones

The mean curvature of a hypersurface at a point is the sum of principal curvatures which is a symmetric function. Similarly, Gauss curvature is the product of the principal curvatures. The study of elementary symmetric functions of principal curvatures will be crucial to analyze the convexity of singularities. We begin by recalling the definition of elementary symmetric polynomials.

Definition 2.1.1. For any $k = 1, \ldots, n$, the **k-th elementary symmetric polynomial** $S_k : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$S_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ with the convention $S_0 \equiv 1$.

Associated to each k we can also define the domain of positivity of first k elementary symmetric polynomials Γ_k given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \ldots, S_k(\lambda) > 0\}$$

It is easy to see that Γ_k are cones in the Euclidean space and satisfy $\Gamma_{k+1} \subset \Gamma_k$. In this formulation a hypersurface is mean-convex if the vector $(\kappa_1, \ldots, \kappa_n)$ is in Γ_1. The following proposition was proved in [HS99a] regarding the cones Γ_k.

Proposition 2.1.1. Let $A = \{x \in \mathbb{R}^n : x_1 > 0, \ldots, x_n > 0\}$ denote the positive cone. The sets Γ_k coincide with the connected component of the domain $\{\lambda \in \mathbb{R}^{n+1} : S_k(\lambda) > 0\}$ containing the positive cone A. Further, the cone Γ_n coincides with the positive cone A.

This establishes a hierarchy of convexity with the last one being uniformly convex where the principal curvature vector $(\kappa_1, \dots, \kappa_n) \in \Gamma_n$ for all points in the hypersurface. The main result of the chapter is the following theorem.

Theorem 2.1.2. Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a smooth solution of the mean curvature flow with $n \geq 2$ such that $X(M^n, 0) = \mathcal{M}_0$ is compact and of positive mean curvature. Then, for any $\eta > 0$ there exists a constant $C_\eta > 0$ depending only on n, η and $\mathcal{M}_0$ such that

$$S_k \geq -\eta H^k - C_{\eta,k} \tag{2.1.1}$$

on $\mathcal{M}_t$ for any $t \in [0, T)$.

This can be interpreted as following - the negative part of S_k becomes very small compared because of the inhomogeneous factor η at points where H^k is large(i.e. where the singularities are developing). We will only prove the theorem for $k = 2$ adapted from [HS99b]. A complete proof is done using induction in [HS99a].

2.2 Estimate of S_2

For any $\eta \in \mathbb{R}$ and $\sigma \in [0, 2]$ let

$$g_{\sigma,\eta} = \left(\frac{|A|^2}{H^2} - (1 + \eta) \right) H^\sigma = \frac{|A|^2 - (1 + \eta)H^2}{H^{2-\sigma}} = \frac{-2S_2 - \eta H^2}{H^{2-\sigma}}.$$

Our aim is to derive a uniform bound of $g_{\sigma,\eta}$ which using Young's inequality will imply the desired estimate. The proof of Theorem 2.1.2 for $k = 2$ is divided into two parts. The first part is obtaining an L^p estimate of $g_{\sigma,\eta}$ and the second part is utilizing Stampacchia lemma using Michael-Simon inequality in order to get an L^∞ bound. In order to prove the first part we derive the evolution equation of $g_{\sigma,\eta}$ using the product rule but before that we need the following lemmas.

Lemma 2.2.1. The following equality holds:

$$|\nabla A \cdot H - \nabla H \otimes A|^2 = |\nabla A|^2 H^2 + |A|^2 |\nabla H|^2 - \left\langle \nabla |A|^2, \nabla H \right\rangle H. \tag{2.2.1}$$

Proof. Computing the norm,

$$\begin{aligned} |\nabla A \cdot H - \nabla H \otimes A|^2 &= \langle \nabla A \cdot H - \nabla H \otimes A, \nabla A \cdot H - \nabla H \otimes A \rangle \\ &= |\nabla A|^2 H^2 + |\nabla H|^2 |A|^2 - 2H \langle \nabla A, \nabla H \otimes A \rangle \\ &= |\nabla A|^2 H^2 + |\nabla H|^2 |A|^2 - \left\langle \nabla |A|^2, \nabla H \right\rangle H. \end{aligned}$$

□

Lemma 2.2.2. The quantity $\dfrac{|A|^2}{H^2}$ satisfies the differential equation

$$\frac{\partial}{\partial t}\frac{|A|^2}{H^2} = \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle \nabla H, \nabla\frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2. \tag{2.2.2}$$

Proof. Computing the time derivative we get

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{|A|^2}{H^2} &= \frac{1}{H^2}\frac{\partial|A|^2}{\partial t} - 2\frac{|A|^2}{H^3}\frac{\partial H}{\partial t}\\
&= \frac{1}{H^2}\left(\Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4\right) - 2\frac{|A|^2}{H^3}\left(\Delta H + |A|^2 H\right)\\
&= \frac{\Delta|A|^2}{H^2} - 2\frac{|\nabla A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3}.
\end{aligned}$$

Recall the division formula for Laplacian,

$$\Delta\left(\frac{u}{v}\right) = \frac{\Delta u}{v} - u\frac{\Delta v}{v^2} - \frac{2}{v^2}\langle\nabla u, \nabla v\rangle + 2\frac{u}{v^3}|\nabla v|^2.$$

Calculating the Laplace-Beltrami operator using this,

$$\begin{aligned}
\Delta\frac{|A|^2}{H^2} &= \frac{\Delta|A|^2}{H^2} - |A|^2\frac{\Delta H^2}{H^4} - \frac{2}{H^4}\left\langle\nabla|A|^2, \nabla H^2\right\rangle + \frac{2|A|^2}{H^6}|\nabla H^2|^2\\
&= \frac{\Delta|A|^2}{H^2} - |A|^2\left(\frac{2H\Delta H + 2|\nabla H|^2}{H^4}\right) - \frac{2}{H^4}\left\langle\nabla|A|^2, 2H\nabla H\right\rangle + 8\frac{|A|^2}{H^6}|\nabla H|^2\\
&= \frac{\Delta|A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3} + 6|A|^2\frac{|\nabla H|^2}{H^4} - \frac{4}{H^3}\left\langle\nabla|A|^2, \nabla H\right\rangle
\end{aligned}$$

which substituted in the time derivative gives

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{|A|^2}{H^2} &= \Delta\frac{|A|^2}{H^2} - 6|A|^2\frac{|\nabla H|^2}{H^4} + \frac{4}{H^3}\left\langle\nabla|A|^2, \nabla H\right\rangle - 2\frac{|\nabla A|^2}{H^2}\\
&= \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle\nabla H, \frac{\nabla|A|^2}{H^2} - \frac{2}{H^3}|A|^2\nabla H\right\rangle\\
&\quad - \frac{2}{H^4}\left(|A|^2|\nabla H|^2 + |\nabla A|^2H^2 - H\left\langle\nabla|A|^2, \nabla H\right\rangle\right)\\
&= \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle\nabla H, \nabla\frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2.
\end{aligned}$$

□

Using this we compute the time derivative of $g_{\sigma,\eta}$.

Lemma 2.2.3. The evolution equation of $g_{\sigma,\eta}$ is given by

$$\begin{aligned}\frac{\partial g_{\sigma,\eta}}{\partial t} =&\Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma(1-\sigma)}{H^2}g_{\sigma,\eta}|\nabla H|^2 \\ &- \frac{2}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta}. \end{aligned} \tag{2.2.3}$$

Proof. We can write $g_{\sigma,\eta} = \left(\frac{|A|^2}{H^2} - (1+\eta)\right)H^\sigma$ so

$$\begin{aligned}\frac{\partial g_{\sigma,\eta}}{\partial t} &= \left\{\Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle \nabla H, \nabla\frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2\right\}H^\sigma \\ &\quad + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\left(\Delta H^\sigma - \sigma(\sigma-1)H^{\sigma-2}|\nabla H|^2 + \sigma|A|^2H^\sigma\right) \\ &= \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\left\langle \nabla H, \nabla\frac{|A|^2}{H^2}\right\rangle H^\sigma - \frac{\sigma(\sigma-1)}{H^2}g_{\sigma,\eta}|\nabla H|^2 \\ &\quad - \frac{2}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta} \\ &= \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\left(\langle\nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma}{H}g_{\sigma,\eta}|\nabla H|^2\right) - \frac{\sigma(\sigma-1)}{H^2}g_{\sigma,\eta}|\nabla H|^2 \\ &\quad - \frac{2}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta} \\ &= \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\langle\nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma(1-\sigma)}{H^2}g_{\sigma,\eta}|\nabla H|^2 \\ &\quad - \frac{2}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta}. \end{aligned}$$

□

Applying the maximum principle on Lemma 2.2.2 gets that $\frac{|A|^2}{H^2}$ is uniformly bounded so there exists a positive constant depending only on $\mathcal{M}_0$ such that

$$|A|^2 \leq \tilde{c_0}H^2 \quad \text{on} \quad \mathcal{M}_t,$$

for all time $t \in [0,T)$. This also implies $g_{\sigma,\eta} \leq c_0 H^\sigma$ but as H blows up this isn't sufficient to prove the uniform bound. The following estimate of the good term in Eq. (2.2.3) will be required for the L^p estimate.

Lemma 2.2.4. [HS99b] If $(1+\eta)H^2 \leq |A|^2 \leq c_0H^2$ for some $\eta, c_0 > 0$. Then

1. $-2Z \geq \eta H^2|A|^2$
2. $|\nabla A\cdot H - \nabla H\otimes A|^2 \geq \frac{\eta^2}{4n(n-1)^2c_0}H^2|\nabla H|^2$

For the rest of proof we will restrict $\eta, \sigma \in (0,1)$ and c_i will denote a constant depending only on n, η and $\mathcal{M}_0$. For brevity, we will write $g = g_{\sigma,\eta}$ as long as σ, η is fixed. Let

$g_+ = \max\{g(x,t), 0\}$ denote the positive part of g. Then $g_+^p \in C^1(\mathcal{M} \times [0,T))$ for $p > 1$ and

$$\partial_t g_+^p = p g_+^{p-1} \partial_t g, \qquad \nabla(g_+^p) = p g_+^{p-1} \nabla g.$$

Lemma 2.2.5. There exists constant c_2, c_3 such that

$$\frac{d}{dt}\int_{\mathcal{M}} g_+^p d\mu \leq -\frac{p(p-1)}{2}\int_{\mathcal{M}} g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int_{\mathcal{M}} \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$- p\int_{\mathcal{M}} \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 d\mu + p\sigma \int_{\mathcal{M}} |A|^2 g_+^p d\mu \quad (2.2.4)$$

for any $p \geq c_2$.

Proof. Differentiating with respect to time and using Lemma 2.2.3 for $p \geq 2$

$$\frac{d}{dt}\int_{\mathcal{M}} g_+^p d\mu = \int \left(p g_+^{p-1}\partial_t g - H^2 g_+^p\right) d\mu$$
$$\leq \int p g_+^{p-1}\left(\Delta g + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g\rangle - \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2\right) d\mu$$
$$+ p\int \sigma |A|^2 g_+^p d\mu \quad (2.2.5)$$

Using integration by parts,

$$\int p g_+^{p-1}\Delta g d\mu = -p\int \left\langle \nabla g_+^{p-1}, \nabla g\right\rangle d\mu$$
$$= -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu \quad (2.2.6)$$

Also from Lemma 2.2.4 we deduce that if $c_1 \geq 4n(n-1)^2 c_0 \eta^{-2}$

$$\frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 \geq \frac{g_+^{p-1}}{c_1 H^{2-\sigma}}|\nabla H|^2$$
$$\geq \frac{g_+^{p-1}}{2c_1 H^{2-\sigma}}|\nabla H|^2 + \frac{1}{2c_0 c_1}\frac{g_+^p}{H^2}|\nabla H|^2 \quad (2.2.7)$$

To handle the gradient term, let $p \geq \max\{2, 1 + 4c_0 c_1\}$ to obtain

$$2(1-\sigma)p\frac{g_+^{p-1}}{H}\langle \nabla H, \nabla g\rangle \leq 2p\frac{g_+^{p-1}}{H}|\nabla H||\nabla g|$$
$$\leq \frac{p}{2c_0 c_1}\frac{g_+^p}{H^2}|\nabla H|^2 + 2c_0 c_1 p g_+^{p-2}|\nabla g|^2 \quad \text{[Peter-Paul inequality]}$$
$$\leq p\frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 - p\frac{g_+^{p-1}}{2c_1 H^{2-\sigma}}|\nabla H|^2$$
$$+ \frac{p(p-1)}{2}g_+^{p-2}|\nabla g|^2 \quad \text{[Using Eq. (2.2.7)]}$$

Substituting this back in Eq. (2.2.5) and using integration by parts from Eq. (2.2.6),

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathcal{M}} g_+^p d\mu \leq & -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu + p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu \\
& + \frac{p(p-1)}{2}\int g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\
& - 2p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu + p\sigma\int |A|^2 g_+^p d\mu
\end{aligned}
$$

which gives the desired inequality with $c_3 = \frac{1}{2c_1}$. □

To handle the bad positive term appearing in Eq. (2.2.4) we use the following lemma

Lemma 2.2.6. There exists a constant c_4 such that

$$
\begin{aligned}
\frac{1}{c_4}\int |A|^2 g_+^p d\mu \leq \left(p + \frac{p}{\beta}\right)\int g_+^{p-2}|\nabla g|^2 + (1+\beta p)\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\
+ \int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu
\end{aligned}
$$

for any $\beta > 0, p > 2$.

Proof. The Laplacian-Beltrami operator satisfies,

$$
\Delta(f^\sigma) = \sigma f^{\sigma-1}\Delta f + \sigma(\sigma-1)f^{\sigma-2}|\nabla f|^2
$$

We have an expression for the Laplacian of $\frac{|A|^2}{H^2}$ in Lemma 2.2.2 from which it follows that

$$
\begin{aligned}
\Delta g &= \Delta\left(\frac{|A|^2}{H^2}\right)H^\sigma + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\Delta H^\sigma + 2\left\langle \nabla\frac{|A|^2}{H^2}, \nabla H^\sigma\right\rangle \\
&= \left(\frac{\Delta|A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3} + 6|A|^2\frac{|\nabla H|^2}{H^4} - \frac{4}{H^3}\left\langle \nabla|A|^2, \nabla H\right\rangle\right)H^\sigma \\
&\quad + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\left(\sigma H^{\sigma-1}\Delta H + \sigma(\sigma-1)H^{\sigma-2}|\nabla H|^2\right) \\
&\quad + 2\sigma H^{\sigma-1}\left\langle \frac{\nabla|A|^2}{H^2} - 2\frac{|A|^2}{H^3}\nabla H, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{|A|^2}{H^{3-\sigma}} - \sigma(1+\eta)H^{\sigma-1}\right)\Delta H + 6\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 - \frac{4}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle \\
&\quad + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2\sigma}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle - 4\sigma\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H + (6-4\sigma)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&\quad - \frac{2}{H^{4-\sigma}}H\left\langle \nabla|A|^2, \nabla H\right\rangle + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H + (6-4\sigma)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&\quad - \frac{2}{H^{4-\sigma}}\left(|\nabla A|^2H^2 + |A|^2|\nabla H|^2 - |\nabla A\cdot H - \nabla H\otimes A|^2\right) + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 \\
&\quad + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2 - 2|\nabla A|^2}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
&\quad - 4(\sigma-1)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle.
\end{aligned}
$$

Now similar to time derivative in Lemma 2.2.3, we calculate inner product of ∇g with ∇H,

$$
\begin{aligned}
\langle \nabla g, \nabla H\rangle &= \left\langle \nabla\frac{|A|^2}{H^2}, \nabla H\right\rangle H^\sigma + \sigma\left(\frac{|A|^2}{H^2} - (1+\eta)\right)H^{\sigma-1}|\nabla H|^2 \\
&= \left\langle \frac{\nabla|A|^2}{H^2}, \nabla H\right\rangle H^\sigma - 2\frac{|A|^2}{H^{3-\sigma}}|\nabla H|^2 + \sigma\frac{g}{H}|\nabla H|^2.
\end{aligned}
$$

Using Simon's identity and the previous expression to eliminate the last mixed inner product term

$$
\begin{aligned}
\Delta g = & \frac{\Delta|A|^2 - 2|\nabla A|^2}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
& - 4(\sigma-1)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 \\
& + \frac{2(\sigma-1)}{H}\left(\langle \nabla g, \nabla H\rangle + 2\frac{|A|^2}{H^{3-\sigma}}|\nabla H|^2 - \sigma\frac{g}{H}|\nabla H|^2\right) \\
= & \frac{2\langle h_{ij}, \nabla_i\nabla_j H\rangle + 2Z}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
& - \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H}\langle \nabla g, \nabla H\rangle
\end{aligned} \tag{2.2.8}
$$

Recall Green's identity for compact manifold without boundary,

$$\int_M u\Delta v = -\int_M \langle \nabla u, \nabla v\rangle .$$

Multiplying Eq. (2.2.8) by $g_+^p H^{-\sigma}$ and using Green's identity the left-hand side evaluates to

$$
\begin{aligned}
A = \int g_+^p H^{-\sigma}\Delta g d\mu = & -\int \left\langle \nabla(g_+^p H^{-\sigma}), \nabla g\right\rangle d\mu \\
& = -p\int \frac{1}{H^\sigma} g_+^{p-1}|\nabla g|^2 d\mu + \sigma\int \frac{g_+^p}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle d\mu
\end{aligned} \tag{2.2.9}
$$

while the right-hand side is

$$
\begin{aligned}
B = & 2\int \frac{\langle h_{ij}, \nabla_i\nabla_j H\rangle g_+^p}{H^2} d\mu + 2\int \frac{g_+^p Z}{H^2} d\mu + 2\int \frac{g_+^p}{H^4}|\nabla A \cdot H - \nabla H \otimes A|^2 d\mu \\
& + (\sigma-2)\int \frac{g_+^{p+1}}{H^{1+\sigma}}\Delta H d\mu - 2(1+\eta)\int \frac{g_+^p}{H}\Delta H d\mu - \sigma(\sigma-1)\int \frac{g_+^{p+1}}{H^{2+\sigma}}|\nabla H|^2 d\mu \\
& + 2(\sigma-1)\int \frac{g_+^{p+1}}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle d\mu
\end{aligned} \tag{2.2.10}
$$

For the first term of Eq. (2.2.10) we can use divergence-type theorem for tensors to get,

$$
\begin{aligned}
2\int \frac{\langle h_{ij}, \nabla_i\nabla_j H\rangle g_+^p}{H^2} d\mu = & -2\int \left\langle \mathrm{tr}_{ik}\left(\nabla_k\left(\frac{g_+^p h_{ij}}{H^2}\right)\right), \nabla_j H\right\rangle d\mu \\
& = -2p\int \frac{g_+^{p-1}}{H^2}\left\langle \nabla^i g \otimes h_{ij}, \nabla_j H\right\rangle d\mu \\
& \quad + 4\int \frac{g_+^p}{H^3}\left\langle \nabla^i H \otimes h_{ij}, \nabla_j H\right\rangle d\mu - 2\int \frac{g_+^p}{H^2}\left\langle \nabla^i h_{ij}, \nabla_j H\right\rangle d\mu
\end{aligned} \tag{2.2.11}
$$

Using Codazzi equation $\nabla^i h_{ij} = \nabla_j h^i_i$ for the last term,

$$2\int \frac{\langle h_{ij}, \nabla_i\nabla_j H\rangle\, g_+^p}{H^2} d\mu = -2p\int \frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i g \nabla_j H\rangle\, d\mu \\ + 4\int \frac{g_+^p}{H^3}\langle h_{ij}, \nabla_i H \nabla_j H\rangle\, d\mu - 2\int \frac{g_+^p}{H^2}|\nabla H|^2 d\mu \quad (2.2.12)$$

Applying Green's formula on ΔH terms in Eq. (2.2.10) and putting together Eq. (2.2.9), Eq. (2.2.10) and Eq. (2.2.12)

$$-p\int \frac{1}{H^\sigma} g_+^{p-1}|\nabla g|^2 d\mu + \underbrace{\sigma\int \frac{g_+^p}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle\, d\mu}_{1}$$

$$= -2p\int \frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i g \nabla_j H\rangle\, d\mu + 4\int \frac{g_+^p}{H^3}\langle h_{ij}, \nabla_i H \nabla_j H\rangle\, d\mu \underbrace{- 2\int \frac{g_+^p}{H^2}|\nabla H|^2 d\mu}_{2}$$

$$+ 2\int \frac{g_+^p Z}{H^2} d\mu + 2\int \frac{g_+^p}{H^4}|\nabla A\cdot H - \nabla H \otimes A|^2 d\mu \underbrace{- (\sigma-2)(p+1)\int \frac{g_+^p}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle\, d\mu}_{1}$$

$$\underbrace{+ (\sigma-2)(1+\sigma)\int \frac{g_+^{p+1}}{H^{2+\sigma}}|\nabla H|^2 d\mu}_{3} + 2(1+\eta)p\int \frac{g_+^{p-1}}{H}\langle \nabla g, \nabla H\rangle\, d\mu$$

$$\underbrace{- 2(1+\eta)\int \frac{g_+^p}{H^2}|\nabla H|^2 d\mu}_{2} \underbrace{- \sigma(\sigma-1)\int \frac{g_+^{p+1}}{H^{2+\sigma}}|\nabla H|^2 d\mu}_{3} \underbrace{+ 2(\sigma-1)\int \frac{g_+^{p+1}}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle\, d\mu}_{1}$$

clubbing the terms with same-numbered under bracket,

$$-2\int \frac{g_+^p Z}{H^2} d\mu = p\int \frac{1}{H^\sigma} g_+^{p-1}|\nabla g|^2 d\mu - 2p\int \frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i g \nabla_j H\rangle\, d\mu \\ + 4\int \frac{g_+^p}{H^3}\langle h_{ij}, \nabla_i H \nabla_j H\rangle\, d\mu + 2\int \frac{g_+^p}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu \\ + p\int \left((2-\sigma)\frac{g_+^p}{H^{1+\sigma}} + 2(1+\eta)\frac{g_+^{p-1}}{H}\right)\langle \nabla g, \nabla H\rangle\, d\mu \\ - 2\int \left(\frac{g_+^{p+1}}{H^{2+\sigma}} + (2+\eta)\frac{g_+^p}{H^2}\right)|\nabla H|^2 d\mu \quad (2.2.13)$$

From Lemma 2.2.4 $-2Z \geq \eta H^2|A|^2$ and using utilizing $g \leq c_0 H^\sigma$ (and $|A| \leq c_0 H$) with Cauchy-Schwarz inequality in Eq. (2.2.13),

$$\eta \int g_+^p |A|^2 d\mu \le c_0 p \int g_+^{p-2} |\nabla g|^2 d\mu + 4p(c_0+1) \int \frac{g_+^{p-1}}{H} |\nabla g||\nabla H| d\mu + 4c_0^2 \int \frac{g_+^{p-1}}{H^{2-\sigma}} |\nabla H|^2 d\mu + 2c_0 \int \frac{g_+^{p-1}}{H^{4-\sigma}} |\nabla A \cdot H - \nabla H \otimes A|^2 d\mu \tag{2.2.14}$$

Also, for any $\beta > 0$,

$$\begin{aligned} 2\frac{g_+^{p-1}}{H}|\nabla H||\nabla g| &\le \frac{g_+^{p-2}}{\beta}|\nabla g|^2 + \beta \frac{g_+^p}{H^2}|\nabla H|^2 \\ &= \frac{g_+^{p-2}}{\beta}|\nabla g|^2 + c_0\beta \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 \end{aligned} \tag{2.2.15}$$

Combining Eq. (2.2.13), Eq. (2.2.14) and Eq. (2.2.15) proves the lemma. □

Proposition 2.2.7. For any $\eta \in (0,1)$ there exists constants c_5, c_6 such that the $L^p(\mathcal{M})$ norm of $(g_{\sigma,\eta})_+$ is a increasing function of t if the following holds

$$p \ge c_5, \qquad \sigma \le (c_6 p)^{-\frac{1}{2}}.$$

Proof. Choose $\beta \sim p^{-\frac{1}{2}}$ and $\sigma \sim cp^{-\frac{1}{2}}$ in the previous lemma. □

Lemma 2.2.8 (Stampacchia lemma)**.** Let $\psi : [k_0, \infty) \to \mathbb{R}$ be a non-negative, non-increasing function which satisfies

$$\psi(h) \le \frac{C}{(h-k)^\alpha}\psi(k)^\beta \text{ for all } h > k > k_0 \tag{2.2.16}$$

for some constants $C > 0$, $\alpha > 0$ and $\beta > 1$. Then

$$\psi(k_0 + d) = 0, \tag{2.2.17}$$

where $d^\alpha = C\psi(k_0)^{\beta-1}2^{\frac{\alpha\beta}{\beta-1}}$.

We complete the proof of Theorem 2.1.2 using Stampacchia lemma which gives an L^∞ bound from the L^p bounds.

Proof. Let $k \ge k_0$, where

$$k_0 = \sup_{\sigma\in[0,1]} \sup_{\mathcal{M}_0} g_{\sigma,\eta}$$

Define $v = (g_{\sigma,\eta} - k)_+^{\frac{p}{2}}$ and $A(k,t) = \{x \in \mathcal{M}_t : v(x,t) > 0\}$. Differentiating v^2 with respect to time we get for p large enough (similar to Lemma 2.2.5)

$$\frac{d}{dt}\int_{\mathcal{M}_t} v^2 d\mu + \int_{\mathcal{M}_t} |\nabla v|^2 d\mu \leq \sigma p \int_{\mathcal{M}_t} |A|^2 v^2 d\mu \leq c_0 \sigma p \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu \tag{2.2.18}$$

Also from the Michael-Simon result in [MS73], we have a Sobolev-type inequality given by

$$\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \leq C(n)\int_{\mathcal{M}_t} |\nabla v|^2 d\mu + C(n)\left(\int_{A(k,t)} H^n d\mu\right)^{\frac{2}{n}}\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \tag{2.2.19}$$

where $q = \frac{n}{n-2}$ if $n > 2$ and an arbitrary number greater than 1 if $n = 2$. We can estimate the H^n factor in the integral on $A(k,t)$ using the previous proposition and the equality

$$\int_{\mathcal{M}_t} H^n g_{\sigma,\eta}^p d\mu = \int_{\mathcal{M}_t} g_{\sigma',\eta}^p d\mu$$

where $\sigma' = \sigma + \frac{n}{p}$. Let

$$p \geq \max\{c_5, 4n^2 c_6\} \quad \text{and} \quad \sigma \leq (4c_6 p^{-\frac{1}{2}})$$

so that

$$\sigma' = \sigma + \frac{n}{p} \leq \frac{1}{2\sqrt{c_6 p}} + \frac{1}{\sqrt{p}}\frac{n}{\sqrt{p}} \leq \frac{1}{\sqrt{c_6 p}}$$

which allows us to use Proposition 2.2.7,

$$\begin{aligned}
\left(\int_{A(k,t)} H^n d\mu\right)^{\frac{2}{n}} &\leq \left(\int_{A(k,t)} H^n \left(\frac{g_{\sigma,\eta}^p}{k^p}\right) d\mu\right)^{\frac{2}{n}} \\
&= k^{-\frac{2p}{n}}\left(\int_{A(k,t)} g_{\sigma',\eta}^p d\mu\right)^{\frac{2}{n}} \\
&\leq k^{-\frac{2p}{n}}\left(\int_{\mathcal{M}_t} (g_{\sigma',\eta})_+^p d\mu\right)^{\frac{2}{n}} \\
&\leq k^{-\frac{2p}{n}}\left(\int_{\mathcal{M}_0} (g_{\sigma',\eta})_+^p d\mu\right)^{\frac{2}{n}} \\
&\leq \left(\frac{|\mathcal{M}_0| k_0}{k}\right)^{\frac{2p}{n}}
\end{aligned}$$

We can fix $k_1 > k_0$ such that for any $k \geq k_1$ the term $\int_{A(k,t)} H^n d\mu$ in Eq. (2.2.19) is less than $\frac{1}{2C(n)}$. For such k, using Eq. (2.2.18) with Eq. (2.2.19) to eliminate the gradient term,

$$\frac{d}{dt}\int_{\mathcal{M}_t} v^2 d\mu + \frac{1}{2C(n)}\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \leq c_0 \sigma p \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu. \tag{2.2.20}$$

Let $t_0 \in [0,T]$ be the time when $\sup_{t\in[0,T)} \int_{\mathcal{M}_t} v^2 d\mu$ is attained (we let $t_0 = T$ if it is not attained in the interior). Integrating Eq. (2.2.20) from 0 to t_0,

$$\int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{2C(n)} \int_0^{t_0} \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q}} dt \leq c_0 \sigma p \int_0^{t_0} \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu dt \tag{2.2.21}$$

where we used the fact that $k > k_0 \geq \sup_{\mathcal{M}_0} g_{\sigma,\eta}$ so $\int_{\mathcal{M}_0} v^2 d\mu = 0$. Now integrating Eq. (2.2.20) from t_0 to $T - \epsilon$ for ϵ small enough,

$$\int_{\mathcal{M}_{T-\epsilon}} v^2 d\mu - \int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{2C(n)} \int_{t_0}^{T-\epsilon} \left(\int_{\mathcal{M}_t} v^{2q} \right)^{\frac{1}{q}} dt \leq c_0 \sigma p \int_{t_0}^{T-\epsilon} \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu dt. \tag{2.2.22}$$

Throwing away $\int_{\mathcal{M}_{T-\epsilon}} v^2 d\mu$ term and adding Eq. (2.2.21) to half of Eq. (2.2.22) after taking the limit $\epsilon \to 0$,

$$\frac{1}{2} \int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{4C(n)} \int_0^T \left(\int_{\mathcal{M}_t} v^{2q} \right)^{\frac{1}{q}} dt \leq c_0 \sigma p \int_0^T \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu dt$$

which is same as

$$\sup_{[0,T)} \int_{\mathcal{M}_t} v^2 d\mu + \int_0^T \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q}} dt \leq 2 \max\{1, 2C(n)\} c_0 \sigma p \int_0^T \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu dt. \tag{2.2.23}$$

Recall the interpolation inequality for L^p spaces for any $f \in L^q \cap L^r$,

$$||f||_{q_0} \leq ||f||_q^{\alpha} ||f||_r^{1-\alpha}$$

where $\frac{1}{q_0} = \frac{\alpha}{q} + \frac{1-\alpha}{q}$ and $1 < q_0 < q$. Setting $r = 1, \alpha = \frac{1}{q_0}$ and $f = v^2$ we get

$$\left(\int_{\mathcal{M}_t} v^{2q_0} d\mu \right)^{\frac{1}{q_0}} \leq \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q_0 q}} \left(\int_{\mathcal{M}_t} v^2 d\mu \right)^{1-\frac{1}{q_0}}. \tag{2.2.24}$$

Integrating this in time and using Young's inequality,

$$\begin{aligned}
\left(\int_0^T \int_{A(k,t)} v^{2q_0} d\mu dt \right)^{\frac{1}{q_0}} &\leq \left(\sup_{[0,T)} \int_{A(k,t)} v^2 d\mu \right)^{1-\frac{1}{q_0}} \left(\int_0^T \left(\int_{A(k,t)} v^{2q} d\mu \right)^{\frac{1}{q}} dt \right)^{\frac{1}{q_0}} \\
&\leq \frac{\sup_{[0,T)} \int_{A(k,t)} v^2 d\mu}{\frac{q_0}{q_0 - 1}} + \frac{\int_0^T \left(\int_{A(k,t)} v^{2q} d\mu \right)^{\frac{1}{q}} dt}{q_0} \\
&\leq \sup_{[0,T)} \int_{A(k,t)} v^2 d\mu + \int_0^T \left(\int_{A(k,t)} v^{2q} d\mu \right)^{\frac{1}{q}} dt \\
&\leq c_8 \sigma p \int_0^T \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu dt
\end{aligned}$$

where $c_8 = 2\max\{1, 2C(n)\}c_0$. Set $\psi(k) = \int_0^T \int_{A(k,t)} d\mu dt$. We will obtain bounds on ψ which along with the Stampacchia lemma will imply a uniform bound of $g_{\sigma,\eta}$. Now Eq. (2.2.23) and Hölder inequality yields,

$$\int_0^T \int_{A(k,t)} v^2 d\mu dt \leq \left(\int_0^T \int_{A(k,t)} 1 d\mu dt\right)^{1-\frac{1}{q_0}} \left(\int_0^T \int_{A(k,t)} v^{2q_0} d\mu dt\right)^{\frac{1}{q_0}} \tag{2.2.25}$$

$$\leq c_8 \sigma p \psi(k)^{1-\frac{1}{q_0}} \int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \tag{2.2.26}$$

Let $r > 1$ which will be chosen later. Applying Hölder again on the right side with weights r and $\frac{r}{r-1}$,

$$\int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \leq \left(\int_0^T \int_{A(k,t)} d\mu dt\right)^{1-\frac{1}{r}} \left(\int_0^T \int_{A(k,t)} H^{2r} g^{pr}_{\sigma,\eta} d\mu dt\right)^{\frac{1}{r}}$$

$$= \psi(k)^{1-\frac{1}{r}} \left(\int_0^T \int_{A(k,t)} g^{pr}_{\sigma'',\eta} d\mu dt\right)^{\frac{1}{r}}$$

where $\sigma'' = \sigma + \frac{2}{p}$. For r large enough and p, σ^{-1} small enough from Proposition 2.2.7 there exists a constant $c_9 > 0$ independent of time such that

$$\int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \leq c_9^{\frac{1}{r}} \psi(k)^{1-\frac{1}{r}}. \tag{2.2.27}$$

Combining Eq. (2.2.26) and Eq. (2.2.27) for all $h > k \geq k_1$, we have

$$\begin{aligned}(h-k)^p \psi(h) &= \int_0^T \int_{A(h,t)} (h-k)^p d\mu dt \\ &\leq \int_0^T \int_{A(k,t)} v^2 d\mu dt \\ &\leq c_8 \sigma p c_9^{\frac{1}{r}} \psi(k)^{2-\frac{1}{r}-\frac{1}{q_0}}.\end{aligned}$$

Let $\gamma = 2 - \frac{1}{r} - \frac{1}{q_0}$ and $c_{10} = c_8 c_9^{\frac{1}{r}}$. Fix $r > \frac{q_0}{q_0-1}$ (so $\gamma > 1$) and p large enough, σ small enough while satisfying the hypothesis of Proposition 2.2.7 such that $\sigma p < 1$ then gives

$$\psi(h) \leq \frac{c_{10}}{(h-k)^p} \psi(k)^\gamma \tag{2.2.28}$$

Stampacchia lemma now implies $\psi(k) = 0$ for all $k \geq k_1 + d$ where $d^p = c_{10} 2^{\frac{\gamma p}{\gamma-1}+1} \psi(k_1)^{\gamma-1}$. Hence,

$$g_{\sigma,\eta} \leq k_1 + d \leq K := k_1 + c_{10} 2^{\frac{\gamma p}{\gamma-1}+1} (|\mathcal{M}_0| T)^{\gamma-1}$$

or

$$|A|^2 - (1+\eta) H^2 \leq K H^{2-\sigma}$$

so by Young's inequality there exists a constant C_η such that,

$$|A|^2 - H^2 \leq \eta H^2 + KH^{2-\sigma} \leq 2\eta H^2 + 2C_\eta.$$

Notice that $|A|^2 - H^2 = -\sum_{i\neq j} \kappa_i\kappa_j = -2S_2$ which implies the desired estimate. □

2.3 Asymptotic convexity

As mentioned in Section 1.6.1, we classify the singularities based on the blow-up rate of $|A|^2$. Recall from maximum principle on Lemma 2.2.2 there exists a c_0 such that $|A|^2 \leq c_0 H^2$ and from algebra we get $H^2 \leq n|A|^2$ so $|A|^2$ and H^2 have same rate of growth. We will focus on the growth of H^2.

The estimates obtained in the previous section will be very useful to obtain an asymptotic analysis of type II singularities. Following [HS99b] suppose a maximal solution $X : M \times [0, T) \to \mathbb{R}^{n+1}$ develops a type II singularity. Choose a sequence of points $\{(x_m, t_m)\}$ in spacetime as follows. For each integer $m \geq 1$, let $t_m \in [0, T - \frac{1}{m}]$, $x_m \in M$ such that

$$H^2(x_m, t_m)\left(T - \frac{1}{m} - t_m\right) = \sup_{(x,t)\in M\times\left[0,T-\frac{1}{m}\right]} H^2(x,t)\left(T - \frac{1}{m} - t\right) \tag{2.3.1}$$

Set $L_m = H(x_m, t_m)$, $\alpha_m = -L_m^2 t_m$ and $\omega_m = L_m^2(T - \frac{1}{m} - t_m)$.

Lemma 2.3.1. For singularities of type II, the following holds as $m \to \infty$,

$$t_m \to T, \quad L_m \to \infty, \quad \alpha_m \to -\infty, \text{ and } \omega_m \to \infty.$$

Proof. Fix $M > 0$. As the singularity is of type II, there exists a $t_M \in [0, T)$ and $x_M \in \mathcal{M}$ such that $H^2(x_M, t_M)(T - t_m) > 2M$. For m large enough we have

$$\bar{t} < T - 1/m, \quad H^2(\bar{x}, \bar{t})(T - \bar{t} - 1/m) > M.$$

It follows

$$\omega_m = H^2\left(x_m, t_m\right)\left(T - t_m - 1/m\right) \geq H^2(\bar{x}, \bar{t})(T - \bar{t} - 1/m) > M.$$

□

Now we will rescale the hypersurfaces to analyze the limiting behavior. For each $m \geq 1$, define a family of immersions by

$$X_m(x,t) = L_m(X(x, L_m^{-2}t + t_m) - X(x_m, t_m)) \text{ for } t \in [\alpha_m, \omega_m].$$

Let A_m and H_m denote the fundamental form of the rescaled immersions. Then by the definition of L_m and X_m we have

$$X_m(x_m, 0) = 0 \quad \text{and} \quad H_m(x_m, 0) = 1.$$

Further, observe that

$$H_m^2(x,t) = L_m^{-2} H^2(x, L_m^{-2} t + t_m) \le \frac{T - \frac{1}{m} - t_m}{T - \frac{1}{m} - t_m - L_m^{-2} t} = \frac{\omega_m}{\omega_m - t}.$$

From the previous lemma $\omega_m \to \infty$, so for any $\epsilon > 0$ and $\overline{\omega}$, there exists a m_0 such that

$$\max_{x \in M} H_m(x,t) \le 1 + \epsilon$$

for any $m \ge m_0$ and $t \in [\alpha_{m_0}, \overline{\omega}]$. Also, observe that the elementary symmetric polynomials of principal curvatures of the indexed hypersurfaces scale as $(S_k)_m = L_m^{-k} S_k$ so

$$\begin{aligned}(S_k)_m &\ge -\eta H_m^k - L_m^{-k} C_{\eta,k} \\ &\ge -\eta(1+\epsilon)^k - L_m^{-k} C_{\eta,k}\end{aligned}$$

which can be made arbitrarily small in the limit $m \to \infty$. The curvature bound implies analogous bounds on the second fundamental form as well as its covariant derivatives. Invoking the Arzela-Ascoli theorem there exists a subsequence of X_k converging uniformly on compact subsets of $\mathbb{R}^{n+1} \times \mathbb{R}$ to a limiting solution X_∞ of the mean curvature flow. This proves the asymptotic convexity of the flow in the following sense.

Theorem 2.3.2. Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth maximal solution of the mean curvature flow with $X(\cdot, 0) = \mathcal{M}_0$ compact and of positive mean curvature. Further, assume that the flow develops a singularity of type II. Then there exists a sequence of rescaled flow $X_k(\cdot, t)$ converging smoothly on every compact set to a mean curvature flow $X_\infty(\cdot, t)$ which is defined for $t \in (-\infty, \infty)$. Also, the limit hypersurface X_∞ is convex (not necessarily uniformly convex) for each $t \in (-\infty, \infty)$ and satisfies $0 < H_\infty \le 1$ everywhere with equality at least at one point.

3 Noncollapsing

Noncollapsing in mean curvature flow is a powerful result that gives a geometric idea about the structure of singularities. It can be used to rule out certain singularity profiles for mean convex mean curvature flow.

3.1 Inscribed curvature

Let $\mathcal{M} \subset \mathbb{R}^{n+1}$ be a smooth hypersurface which is the boundary of an open set $\Omega \subset \mathbb{R}^{n+1}$. For $x \in \mathcal{M}$, we want to find the radius of the largest inscribed sphere in $\mathcal{M}$ touching it at x. For any $y \in \mathcal{M}\backslash\{x\}$, the radius of the sphere passing through x and y and touching $\mathcal{M}$ at x is given by

$$r(x,y) = \frac{||x-y||^2}{2\langle x-y, \nu(x)\rangle} \tag{3.1.1}$$

where $\nu(x)$ is the outward unit normal vector of $\mathcal{M}$ at x. The inverse of the radius is the **extrinsic ball curvature** $k : \mathcal{M} \times \mathcal{M}\backslash\{(x,x) : x \in \mathcal{M}\} \to \mathbb{R}$ defined by

$$k(x,y) = \frac{2\langle x-y, \nu(x)\rangle}{||x-y||^2}. \tag{3.1.2}$$

Now for each point $x \in \mathcal{M}$, we can get the radius of the largest inscribed sphere touching $\mathcal{M}$ at x which we call the **inradius** function $r : \mathcal{M} \to (0, \infty]$ given by

$$r(x) = \inf_{y\in\mathcal{M}\backslash\{x\}} r(x,y). \tag{3.1.3}$$

Similarly, the **inscribed curvature** $k : \mathcal{M} \to [0, \infty)$ is obtained by the reciprocal of the inradius, so

$$k(x) = \frac{1}{r(x)} = \sup_{y\in\mathcal{M}\backslash\{x\}} k(x,y). \tag{3.1.4}$$

Definition 3.1.1. Let $\mathcal{M}$ be a mean convex hypersurface bounding an open set $\Omega \subset \mathbb{R}^{n+1}$, so $\partial\Omega = \mathcal{M}$. We say that $\mathcal{M}$ is α-**noncollapsed** if for every $x \in \mathcal{M}$ there exists an open ball B of radius $\frac{\alpha}{H(x)}$ touching $\mathcal{M}$ at x and contained entirely in Ω. In terms of the inscribed curvature, this is same as the inequality

$$k(x) \leq \frac{1}{\alpha} H(x) \quad \text{for all} \quad x \in \mathcal{M}. \tag{3.1.5}$$

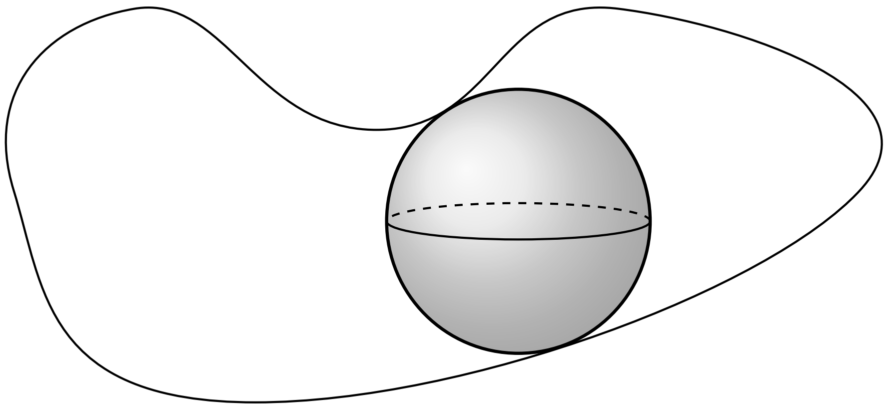

Figure 3.1: Inscribed sphere of maximum radius

We will prove in the following section that noncollapsing is preserved under mean curvature flow.

3.2 Differential inequality for inscribed curvature

Following [ACGL22, Bre15] the time evolution equation of inscribed curvature satisfies an inequality which implies noncollapsing. The main difficulty of the proof lies in manipulating the complicated time derivative of k to get the useful gradient terms with signs.

Theorem 3.2.1. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth solution of the mean curvature flow with $\mathcal{M}_0$ properly embedded. Then

$$\frac{\partial k}{\partial t} \leq \Delta k + |A|^2 k - 2 \sum_{\kappa_i < k} \frac{(\nabla_i k)^2}{k - \kappa_i} \tag{3.2.1}$$

where the inequality holds in the viscosity sense.

Proof. For any given point $(x_0, t_0) \in \mathcal{M} \times [0,T)$, either of the two cases occur

1. $k(x_0,t_0) = \lim_{y \to x_0} k(x_0, y, t_0)$, or
2. $k(x_0,t_0) = k(x_0, y_0, t_0)$ for some $y_0 \in \mathcal{M}_{t_0} \backslash \{x_0\}$.

We will be concentrating only on the second case which happens on an open subset of spacetime, where the supremum is achieved from a sphere touching the hypersurface at two separate points. The first case occurs on a set with measure zero and is covered in Proposition 12.8 in [ACGL22].

Let U be an open neighborhood of x_0 and $\psi : U \times (t_0 - \alpha, t_0] \to \mathbb{R}$ be a smooth function such that $\psi(x_0, t_0) = k(x_0, t_0)$ and $\psi(x, t) \geq k(x, t)$ for all $(x, t) \in U \times (t_0 - \alpha, t_0]$. We want to prove

$$\frac{\partial \psi}{\partial t} \leq \Delta \psi + |A|^2 \psi - 2 \sum_{i=1}^{n} \frac{(\nabla_i \psi)^2}{\psi - \kappa_i}. \tag{3.2.2}$$

Define

$$Z(x, y, t) = \frac{1}{2} \psi(x, t) |X(x, t) - X(y, t)|^2 - \langle X(x, t) - X(y, t), \nu(x, t) \rangle \tag{3.2.3}$$

which can be further simplified to

$$Z(x, y, t) = \frac{|X(x, t) - X(y, t)|^2}{2} (\psi(x, t) - k(x, t))$$

for all $x \neq y$. Observe that $Z(x_0, y_0, t_0) = 0$ and $Z(x, y, t) \geq 0$ for all $(x, y, t) \in U \times M \times (t_0 - \alpha, t_0]$ from the hypothesis on ψ.

The space derivatives in local coordinates are

$$\begin{aligned} \frac{\partial Z}{\partial x_i} &= \frac{1}{2} \frac{\partial \psi}{\partial x_i}(x, t) |X(x, t) - X(y, t)|^2 + \psi(x, t) \left\langle X(x, t) - X(y, t), \frac{\partial X}{\partial x_i}(x, t) \right\rangle \\ &\quad - h_i^k(x, t) \left\langle X(x, t) - X(y, t), \frac{\partial X}{\partial x_k}(x, t) \right\rangle \end{aligned} \tag{3.2.4}$$

and

$$\frac{\partial Z}{\partial y_i} = -\psi(x, t) \left\langle \frac{\partial X}{\partial y_i}(y, t), X(x, t) - X(y, t) \right\rangle + \left\langle \frac{\partial X}{\partial y_i}(y, t), \nu(x, t) \right\rangle \tag{3.2.5}$$

Choose normal coordinates around x_0 such that $h_{ij}(x_0, t_0)$ is diagonal. Then Eq. (3.2.4) at the minima (x_0, y_0) gives

$$\left\langle \eta, \frac{\partial X}{\partial x_i}(x_0, t_0) \right\rangle = -\frac{1}{2} \frac{1}{\psi(x_0, t_0) - \kappa_i(x_0, t_0)} \frac{\partial \psi}{\partial x_i}(x_0, t_0) |X(x_0, t_0) - X(y_0, t_0)| \tag{3.2.6}$$

where $\eta = \frac{X(x_0,t_0) - X(y_0,t_0)}{|X(x_0,t_0) - X(y_0,t_0)|}$. The tangent space at y_0 can be obtained by reflection across the hyperplane with normal η. In particular,

$$\begin{aligned} \nu(y_0, t_0) &= \nu(x_0, t_0) - 2\eta \langle \nu(x_0, t_0), \eta \rangle \\ &= \nu(x_0, t_0) - \psi(x_0, t_0)(X(x_0, t_0) - X(y_0, t_0)). \end{aligned} \tag{3.2.7}$$

Now we choose normal coordinates around y_0, such that the reflection of $\frac{\partial X}{\partial x_i}$ across η is $\frac{\partial X}{\partial y_i}$, so

$$\frac{\partial X}{\partial y_i} = \frac{\partial X}{\partial x_i} - 2\eta \left\langle \frac{\partial X}{\partial x_i}, \eta \right\rangle \tag{3.2.8}$$

Also,

$$\left\langle \frac{\partial X}{\partial y_i}(y_0, t_0), \frac{\partial X}{\partial x_i}(x_0, t_0) \right\rangle = 1 - 2\eta \left\langle \frac{\partial X}{\partial x_i}(x_0, t_0), \eta \right\rangle^2 \tag{3.2.9}$$

Further, calculating the double space derivatives,

$$\begin{aligned}\frac{\partial^2 Z}{\partial x_i^2}(x, y, t) = & \frac{1}{2}\frac{\partial^2 \psi}{\partial x_i^2}|X(x,t) - X(y,t)|^2 + 2\frac{\partial \psi}{\partial x_i}\left\langle X(x,t) - X(y,t), \frac{\partial X}{\partial x_i}\right\rangle \\ & + \psi \left\langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_i}\right\rangle + \psi \left\langle X(x,t) - X(y,t), \frac{\partial^2 X}{\partial x_i^2}\right\rangle \\ & - \frac{\partial h_i^k}{\partial x_i}\left\langle X(x,t) - X(y,t), \frac{\partial X}{\partial x_k}\right\rangle - h_i^k \left\langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_k}\right\rangle \\ & - h_i^k \left\langle X(x,t) - X(y,t), \frac{\partial^2 X}{\partial x_i \partial x_k}\right\rangle . \end{aligned} \tag{3.2.10}$$

Recall we had chosen normal coordinates at x_0 such that the matrix $h_{ij}(x_0, t_0)$ is diagonal so

$$\frac{\partial^2 X}{\partial x_i^2} = \Gamma_{ii}^k \frac{\partial X}{\partial x_k} - h_{ii}\nu = -\kappa_i \nu.$$

Adding the Eq. (3.2.10) from $i = 1$ to n, and evaluating it at (x_0, y_0, t_0),

$$\begin{aligned}\sum_{i=1}^n \frac{\partial^2 Z}{\partial x_i^2}(x_0, y_0, t_0) = & \frac{1}{2}\Delta\psi|X(x_0,t_0) - X(y_0,t_0)|^2 - 2\sum_{i=1}^n \frac{\partial \psi}{\partial x_i}(x_0,t_0)\left\langle X(x_0,t_0) - X(y_0,t_0), \frac{\partial X}{\partial x_i}\right\rangle \\ & + n\psi + \psi \left\langle X(x_0,t_0) - X(y_0,t_0), -H(x_0,t_0)\nu(x_0,t_0)\right\rangle \\ & - \sum_{i=1}^n \frac{\partial H}{\partial x_i}(x_0,t_0)\left\langle X(x_0,t_0) - X(y_0,t_0), \frac{\partial X}{\partial x_i}\right\rangle - H(x_0,t_0) \\ & + |A(x_0,t_0)|^2 \left\langle X(x_0,t_0) - X(y_0,t_0), \nu(x_0,t_0)\right\rangle \end{aligned} \tag{3.2.11}$$

where we used the Codazzi equation $\sum \partial_i h_i^k = \partial_k H$ for normal coordinates and the mean curvature vector equation $\Delta X = -H\nu$. Using Eq. (3.2.6) this can be written as,

$$\begin{aligned}\sum_{i=1}^n \frac{\partial^2 Z}{\partial x_i^2}(x_0, y_0, t_0) = & \frac{1}{2}\Bigg(\Delta\psi(x_0,t_0) + |A(x_0,t_0)|^2\psi(x_0,t_0) \\ & - \sum_{i=1}^n \frac{2}{\psi(x_0,t_0) - \kappa_i(x_0,t_0)}\left(\frac{\partial \psi}{\partial x_i}(x_0,t_0)\right)^2\Bigg)|X(x_0,t_0) - X(y_0,t_0)|^2 \\ & - \sum_{i=1}^n \frac{\partial H}{\partial x_i}(x_0,t_0)\left\langle X(x_0,t_0) - X(y_0,t_0), \frac{\partial X}{\partial x_i}(x_0,t_0)\right\rangle \\ & - H(x_0,t_0)\psi(x_0,t_0)\left\langle X(x_0,t_0) - X(y_0,t_0), \nu(x_0,t_0)\right\rangle \\ & + n\psi(x_0,t_0) - H(x_0,t_0). \end{aligned} \tag{3.2.12}$$

Now the mixed derivatives are given by

$$\begin{aligned}\frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y,t) = &-\frac{\partial \psi}{\partial x_i}(x,t)\left\langle \frac{\partial X}{\partial y_i}(y,t), X(x,t) - X(y,t)\right\rangle - \psi(x,t)\left\langle \frac{\partial X}{\partial y_i}(y,t), \frac{\partial X}{\partial x_i}(x,t)\right\rangle \\ &+ \left\langle \frac{\partial X}{\partial y_i}(y,t), \frac{\partial \nu}{\partial x_i}(x,t)\right\rangle.\end{aligned}$$

Evaluating this at (x_0, y_0, t_0) and using Eq. (3.2.6) and Eq. (3.2.9), we get

$$\begin{aligned}\frac{\partial^2 Z}{\partial x_i \partial y_i}(x_0,y_0,t_0) &= -\frac{\partial \psi}{\partial x_i}(x_0,t_0)\left\langle \frac{\partial X}{\partial y_i}(y_0,t_0), X(x_0,t_0) - X(y_0,t_0)\right\rangle \\ &\quad - (\psi(x_0,t_0) - \kappa_i(x_0,t_0))\left\langle \frac{\partial X}{\partial y_i}(y_0,t_0), \frac{\partial X}{\partial x_i}(x_0,t_0)\right\rangle \\ &= \frac{\partial \psi}{\partial x_i}(x_0,t_0)\left\langle X(x_0,t_0) - X(y_0,t_0), \frac{\partial X}{\partial x_i}(x_0,t_0)\right\rangle \\ &\quad - (\psi(x_0,t_0) - \kappa_i(x_0,t_0))\left(1 - 2\eta\left\langle \frac{\partial X}{\partial x_i}(x_0,t_0)\right\rangle^2\right) \\ &= -(\psi(x_0,t_0) - \kappa_i(x_0,t_0)).\end{aligned}$$

For the second order y derivative,

$$\begin{aligned}\frac{\partial^2 Z}{\partial y_i^2}(x,y,t) = &-\psi(x,t)\left\langle \frac{\partial^2 X}{\partial y_i^2}(y,t), X(x,t) - X(y,t)\right\rangle - \psi(x,t)\left\langle \frac{\partial X}{\partial y_i}(y,t), -\frac{\partial X}{\partial y_i}(y,t)\right\rangle \\ &+ \left\langle \frac{\partial^2 X}{\partial y_i^2}(y,t), \nu(x,t)\right\rangle\end{aligned}$$

so from Eq. (3.2.7) at (x_0, y_0, t_0),

$$\begin{aligned}\frac{\partial Z}{\partial y_i^2}(x_0,y_0,t_0) &= \psi(x_0,t_0)\kappa_i(y_0,t_0)\left\langle X(x_0,t_0) - X(y_0,t_0), \nu(y_0,t_0)\right\rangle + \psi(x_0,t_0) \\ &\quad - \kappa_i(y_0,t_0)\left\langle \nu(y_0,t_0), \nu(x_0,t_0)\right\rangle \\ &= \psi(x_0,t_0)\kappa_i(y_0,t_0)\left(\left\langle X(x_0,t_0) - X(y_0,t_0), \nu(x_0,t_0)\right\rangle - \psi(x_0,t_0)|X(x_0,t_0) - X(y_0,t_0)|^2\right) \\ &\quad + \psi(x_0,t_0) - \kappa_i(y_0,t_0)(1 - \psi(x_0,t_0)\left\langle X(x_0,t_0) - X(y_0,t_0), \nu(x_0,t_0)\right\rangle) \\ &= \psi(x_0,t_0) - \kappa_i(y_0,t_0). \qquad (3.2.13)\end{aligned}$$

Now the time derivative is,

$$
\begin{aligned}
\frac{\partial Z}{\partial t}(x_0, y_0, t_0) = & \frac{1}{2}\frac{\partial \psi}{\partial t}(x_0, t_0)|X(x_0, t_0) - X(y_0, t_0)|^2 \\
& + \psi(x_0, t_0)\left\langle -H(x_0, t_0)\nu(x_0, t_0) + H(y_0, t_0)\nu(y_0, t_0), X(x_0, t_0) - X(y_0, t_0)\right\rangle \\
& - \left\langle -H(x_0, t_0)\nu(x_0, t_0) + H(y_0, t_0)\nu(y_0, t_0), \nu(x_0, t_0)\right\rangle \\
& - \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}(x_0, t_0)\left\langle X(x_0, t_0) - X(y_0, t_0), \frac{\partial X}{\partial x_i}(x_0, t_0)\right\rangle \\
= & \frac{1}{2}\frac{\partial \psi}{\partial t}(x_0, t_0)|X(x_0, t_0) - X(y_0, t_0)|^2 \\
& - \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}(x_0, t_0)\left\langle X(x_0, t_0) - X(y_0, t_0), \frac{\partial X}{\partial x_i}(x_0, t_0)\right\rangle \\
& - H(x_0, t_0)\psi(x_0, t_0)\left\langle X(x_0, t_0) - X(y_0, t_0), \nu(x_0, t_0)\right\rangle + H(x_0, t) - H(y_0, t_0)
\end{aligned}
$$

where $\nu(x_0, t_0) - \psi(x_0, t_0)(X(x_0, t_0) - X(y_0, t_0)) = \nu(y_0, t_0)$ was used for last equality. Putting together we get the elliptic term,

$$
\begin{aligned}
& \frac{\partial Z}{\partial t}(x_0, y_0, t_0) - \sum_{i=1}^{n}\left(\frac{\partial^2 Z}{\partial x_i^2} + 2\frac{\partial^2 Z}{\partial x_i \partial y_i} + \frac{\partial^2 Z}{\partial y_i^2}\right)(x_0, y_0, t_0) \\
& = \frac{1}{2}\Bigg(\frac{\partial \psi}{\partial t}(x_0, t_0) - \Delta\psi(x_0, t_0) - |A(x_0, t_0)|^2\psi(x_0, t_0) \\
& \quad + \sum_{i=1}^{n} \frac{2}{\psi(x_0, t_0) - \kappa_i(x_0, t_0)}\left(\frac{\partial \psi}{\partial x_i}(x_0, t_0)\right)^2\Bigg)|X(x_0, t_0) - X(y_0, t_0)|^2.
\end{aligned}
$$

As (x_0, y_0, t_0) is a local minimum of Z, the left-hand side of the previous equation is negative from which the inequality Eq. (3.2.2) follows. □

Remark. Notice that the inequality is in one variable however, the proof goes through the maximum principle on a two-variable function. There are a lot of applications of such two-point functions considered in [Bre14, And14].

Corollary (Noncollapsing)**.** Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a smooth embedded solution of the mean curvature with $X(\cdot, 0) = \mathcal{M}_0$ compact, mean-convex and α-noncollapsed. Then $X(\cdot, t) = \mathcal{M}_t$ is α-noncollapsed for all $t \in [0, T)$.

Proof. From Eq. (3.2.1), it follows that

$$
\frac{\partial k}{\partial t} \leq \Delta k + |A|^2 k.
$$

Recall that the mean curvature H, satisfies Eq. (1.3.4) so the time derivative of the quotient $\frac{k}{H}$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{k}{H}\right) &\leq \frac{(\Delta k + |A|^2 k)H - (\Delta H + |A|^2 H)k}{H^2} \\ &= \Delta\left(\frac{k}{H}\right) + \frac{2}{H}\left\langle \nabla\left(\frac{k}{H}\right), \nabla H\right\rangle.\end{aligned}$$

Now maximum principle (for viscosity solutions) yields $\alpha k \leq H$ for all $t \in [0, T)$.

□

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