Existence of a Willmore Torus

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Setup

- For an embedded surface $\Sigma \subset \mathbb{R}^n$, define the Willmore functional as

$$\mathcal{F}(\Sigma) = \frac{1}{4} \int_{\Sigma} |\mathbf{H}|^2 d\mathcal{H}^2,$$

where $\mathbf{H} = \mathbf{A}(e_1, e_1) + \mathbf{A}(e_2, e_2)$ is the mean curvature.

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- Willmore functional is invariant under scaling, translation, and conformal transformations of $\mathbb{R}^n$.
- For surfaces Σ with genus g and $n \geq 3$ fixed, let

$$\beta_g^n = \inf \mathcal{F}(\Sigma).$$

Then

$$4\pi \leq \beta_g^n < 8\pi.$$

Main theorem

Theorem (Simon, 1993)

For any $n \geq 3$, there exists a smooth real analytic embedded torus in $\mathbb{R}^n$ minimizing the Willmore functional among all embedded tori in $\mathbb{R}^n$.

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Theorem (Bauer-Kuwert, 2003)

For any $n \geq 3$ and genus g, there exists a smooth embedded genus g surface minimizing the Willmore functional among all embedded surfaces of genus g in $\mathbb{R}^n$.

Roadmap

Diameter estimates

Graphical decomposition lemma

Measure theoretic limit
of a minimizing sequence is analytic (possibly a sphere)

Conformally changing the sequence such that the limit is a torus

Diameter estimates

We will first establish upper and lower bounds on the diameter of a surface in terms of its area and Willmore energy.

Lemma

If $\partial\Sigma = \phi$ and Σ is connected, then

$$\sqrt{\frac{|\Sigma|}{\mathcal{F}(\Sigma)}} \leq \mathrm{diam}(\Sigma) \leq C\sqrt{|\Sigma|\mathcal{F}(\Sigma)}.$$

Proof

The idea is to use divergence formula. For a C^1 vector field $\Phi = (\Phi_1, ..., \Phi_n)$,

$$\int_\Sigma \operatorname{div}_\Sigma \Phi = -\int_\Sigma \Phi \cdot \mathbf{H}.$$

For $y \in \Sigma$ and $\Phi = x - y$ with $\operatorname{div}_\Sigma \Phi = 2$,

$$2|\Sigma| = -\int_\Sigma (x-y) \cdot \mathbf{H} \leq 2 \operatorname{diam}(\Sigma) \sqrt{|\Sigma|} \sqrt{\mathcal{F}(\Sigma)}.$$

Proof

For $0 < \sigma < \rho$, let $X = x - y$ and consider the vector field $\Phi = (\max\{|X|, \sigma\}^{-2} - \rho^{-2})_+ X$ to get

$$\begin{aligned} 2\sigma^{-2}|\Sigma_\sigma| - 2\rho^{-2}|\Sigma_\rho| + 2\int_{\Sigma_{\sigma,\rho}} \frac{|X^\perp|^2}{|X|^4} &= -\int_{\Sigma_\rho} (\max\{|X|, \sigma\}^{-2} - \rho^{-2}) X \cdot \mathbf{H} \\ &= -\int_{\Sigma_\sigma} \sigma^{-2} X \cdot \mathbf{H} - \int_{\Sigma_{\sigma,\rho}} \frac{X}{|X|^2} \cdot \mathbf{H} \\ &\quad + \int_{\Sigma_\rho} \rho^{-2} X \cdot \mathbf{H}. \end{aligned}$$

where $\Sigma_r = \Sigma \cap B_r(y)$ and $\Sigma_{\sigma,\rho} = \Sigma_\rho \setminus \Sigma_\sigma$.

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where $\Sigma_r = \Sigma \cap B_r(y)$ and $\Sigma_{\sigma,\rho} = \Sigma_\rho \setminus \Sigma_\sigma$. Completing the square with

$$\frac{|X^\perp|^2}{|X|^4} + \frac{1}{2}\frac{X}{|X|^2} \cdot \mathbf{H} = \left|\frac{1}{4}\mathbf{H} + \frac{X^\perp}{|X|^2}\right|^2 - \frac{1}{16}|\mathbf{H}|^2,$$

Proof

We get

$$\sigma^{-2}|\Sigma_\sigma| + \int_{\Sigma_{\sigma,\rho}} \left|\frac{1}{4}\mathbf{H} + \frac{X^\perp}{|X|^2}\right|^2 = \rho^{-2}|\Sigma_\rho| + \frac{1}{4}\mathcal{F}(\Sigma_{\sigma,\rho}) + \frac{1}{2}\int_{\Sigma_\rho} \rho^{-2} X\cdot\mathbf{H} - \frac{1}{2}\int_{\Sigma_\sigma} \sigma^{-2} X\cdot\mathbf{H}. \tag{1}$$

Let

$$\gamma(r) = r^{-2}|\Sigma_r| + \frac{1}{4}\mathcal{F}(\Sigma_r) + \frac{1}{2}\int_{\Sigma_r} r^{-2} X\cdot\mathbf{H},$$

then eq. (1) rearranges to

$$\gamma(\sigma) + \int_{\Sigma_{\sigma,\rho}} \left|\frac{1}{4}\mathbf{H} + \frac{X^\perp}{|X|^2}\right|^2 = \gamma(\rho)$$

which implies that $\gamma(\rho)$ is monotonically nondecreasing in ρ.

Proof

$$\gamma(r) = r^{-2}|\Sigma_r| + \frac{1}{4}\mathcal{F}(\Sigma_r) + \frac{1}{2}\int_{\Sigma_r} r^{-2} X \cdot \mathbf{H},$$

Note that $\lim_{\sigma\to 0}\gamma(\sigma) = \pi$ and $\lim_{\rho\to\infty}\gamma(\rho) = \frac{1}{4}\mathcal{F}(\Sigma)$ which reproves $4\pi \leq \mathcal{F}(\Sigma)$. One can also prove that equality holds if and only if Σ is a round sphere.

The above can also be generalized to varifolds leading to Li-Yau inequality $4\pi\theta^2(y) \leq \mathcal{F}(\Sigma)$ where $\theta^2(y)$ is the density of the varifold at y as done by Kuwert-Schätzle.

Proof

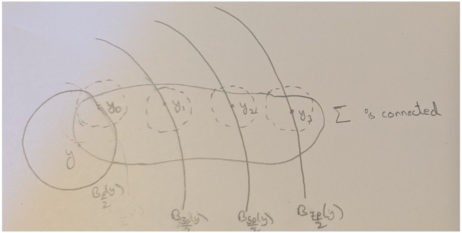
The last term in γ can be estimated by Cauchy-Schwarz,

$$\frac{1}{2}\int_{\Sigma_r} r^{-2} X \cdot \mathbf{H} \le \frac{1}{4}\int_{\Sigma_r} r^{-4}|X|^2 + |\mathbf{H}|^2 \le \frac{r^{-2}|\Sigma_r| + \mathcal{F}(\Sigma_r)}{4}.$$

Together with monotonicity and $\gamma(0) = \pi$, we have

$$\pi = \gamma(0) \le \gamma(\rho) = \frac{5}{4}\rho^{-2}|\Sigma_\rho| + \frac{1}{2}\mathcal{F}(\Sigma_\rho) \le \frac{5}{4}(\rho^{-2}|\Sigma_\rho| + \mathcal{F}(\Sigma_\rho)). \qquad (2)$$

Let $d = \mathrm{diam}(\Sigma)$ and $\rho \in (0, \frac{d}{2}]$. Let $N = \lfloor \frac{d}{\rho} \rfloor \ge 2$. For each $j = 0, \ldots, N-1$ take $y_j \in \partial B_{(j+\frac{1}{2})\rho}(y)$ with $y_0 = y$. The balls, $B_{\frac{\rho}{2}}(y_j)$ are pairwise disjoint.



Proof

By eq. (2) for each y_j and summing over we get

$$N\pi \leq \frac{5}{4}\left(\rho^{-2}|\Sigma| + \mathcal{F}(\Sigma)\right).$$

Select $\rho = \frac{1}{4}\sqrt{|\Sigma|/\mathcal{F}(\Sigma)}$ and use $\frac{d}{2\rho} \leq N$ to get

$$\frac{2d\sqrt{\mathcal{F}(\Sigma)}}{\sqrt{|\Sigma|}} = \frac{d\pi}{2\rho} \leq N\pi \leq \frac{85}{4}\mathcal{F}(\Sigma)$$

which gives the upper bound on diameter.

Graphical decomposition lemma L^1 bound

Let Σ be a smooth compact surface embedded in $\mathbb{R}^n$. For any $\beta > 0$, there is an $\epsilon_0 = \epsilon_0(n, \beta) > 0$ (independent of Σ, ρ) such that if $\epsilon \in (0, \epsilon_0]$, $\partial\Sigma \cap \bar{B}_\rho = \varnothing$, $0 \in \Sigma$, $|\bar{\Sigma}_\rho| \leq \beta\rho^2$, and $\int_{\Sigma_\rho} |\mathbf{A}| \leq \epsilon\rho$, then the following holds:
There are pairwise disjoint closed sets $P_1, \ldots, P_N \subset \Sigma$ with

$$\sum_j \operatorname{diam} P_j \leq C\epsilon^{\frac{1}{2}}\rho \quad \text{and} \quad \Sigma \cap B_{\frac{\rho}{2}} \setminus \left(\bigcup_{i=1}^{N} P_i\right) = \left(\bigcup_{i=1}^{M} \operatorname{graph} u_i\right) \cap B_{\frac{\rho}{2}},$$

where each $u_i \in C^\infty(\Omega_i, L_i^\perp)$ is a k_i valued function for $k_i \geq 1$ ($k_i = 1\,\forall i$ if $n = 3$), L_i is a plane in $\mathbb{R}^n$, and $\Omega_i \subset L_i$ is a domain. Moreover,

$$\sum_{i=1}^{M} k_i \leq C\beta, \quad \text{and} \quad \sup_{\Omega_i} \frac{|u_i|}{\rho} + \sup_{\Omega_i} |Du_i| \leq C\epsilon^{\frac{1}{2(2n-3)}}.$$

Graphical decomposition lemma L^2 bound

If $\int_{\Sigma_\rho} |\mathbf{A}|^2 \leq \epsilon^2$, then the following holds:
All $k_i = 1$ and for any $\sigma \in (\frac{\rho}{4}, \frac{\rho}{2})$ such that ∂B_σ intersects Σ transversely and $\partial B_\sigma \cap (\cup_j P_j) = \varnothing$, we have

$$\Sigma \cap \bar{B}_\sigma = \cup_{i=1}^{M} D_{\sigma,i},$$

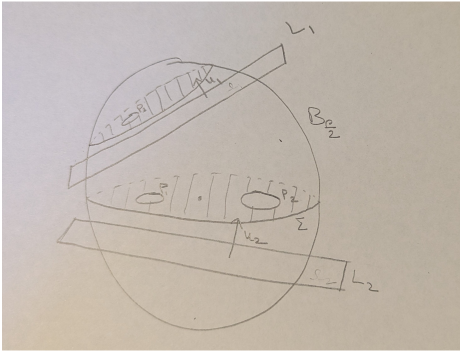
where each $D_{\sigma,i}$ is topologically a disk with $\mathrm{graph}\, u_i \cap \bar{B}_\sigma \subset D_{\sigma,i}$ and

$$D_{\sigma,i} \setminus \mathrm{graph}\, u_i$$

is a union of a subcollection of the P_j, and each P_j is topologically a disk.

Graphical decomposition lemma

The above lemma decomposes $\Sigma \cap B_\sigma$ into a union of disks with smooth boundary contained in ∂B_σ. Each of the disk is a graph of smooth function with small gradient except at few small holes (the pimple P_i's).



Sketch of proof

The proof can be divided into the following steps:

1. Finding planes where surface is graph with small gradient.
2. Picking domains and pimples.
3. Bounds on number of graphical pieces and diameter of pimples.
4. L^2 curvature bound case.

Finding planes where surface is graphical with small gradient

Let

$$S = \{\tau_1 \wedge \tau_2 : \tau_1, \tau_2 \in \mathbb{S}^{n-1}, \tau_1 \cdot \tau_2 = 0\}$$

is the Grassmannian of planes in $\mathbb{R}^n$ of dimension $2n-4$. Cover S with balls $B_{\frac{\eta}{2}}(\tau_j)$ for $j = 1, \ldots, M$ with

$$M \leq \frac{C(n)}{\eta^{2n-4}}.$$

For $x \in \Sigma$, let $\tau(x) \in S$ denote the tangent plane of Σ at x. For any $\tau_0 \in S$, consider the function

$$f_i(x) = |\tau(x) - \tau_i|, \qquad x \in \Sigma_\rho.$$

The gradient of f can be estimated as $|\nabla f| \leq |\mathbf{A}|$.

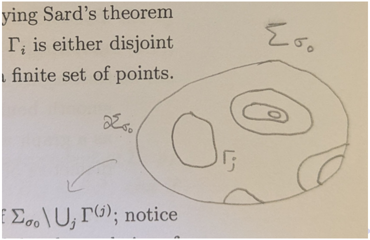
Finding planes where surface is graphical with small gradient

Let $\Gamma^i_t = \{x \in \Sigma_{\rho_0} : f_i(x) = t\}$. Then by Sard's theorem, there is a $t_i \in (\eta, 2\eta)$ such that $\Gamma^i_{t_i}$ is a union of finitely many disjoint Jordan curves and Jordan arcs, the endpoints of the arcs being in ∂B_{σ_0}. Also choosing t_i's successively with Sard's theorem, we can ensure that the Jordan curves in $\Gamma^i_{t_i}$ are either disjoint or intersect transversely.

Let Q_j be the connected components of $\Sigma_{\sigma_0} \setminus \cup_{i=1}^M \Gamma^i_{t_i}$. Then by construction,

$$\sup |\tau(x) - \tau(y)| \leq 2\eta, \quad x, y \in Q_j.$$

For each Q_j, pick a y_j in the interior and let L_j be the plane passing through y_j with orientation $\tau(y_j)$.



Regularity of minimizing sequence

Theorem

Let Σ_k be a sequence of smooth embedded surfaces in $\mathbb{R}^n$ with $|\Sigma_k| = 1$, $0 \in \Sigma_k$ and

$$\mathcal{F}(\Sigma_k) \to \beta_g^n.$$

Then there is a subsequence converging to a smooth embedded surface Σ of genus g_0 with $g_0 \leq g$ which is a minimizer in genus g_0 class.

Double sheets don't minimize Willmore functional

Simon proves a variant of Li-Yau inequality which prevent double sheeting behaviour of the minimizing sequence.

Lemma

Let Σ be a compact surface without boundary, ∂B_ρ intersects Σ transversely, and $\Sigma \cap B_\rho$ contains disjoint subsets Σ_1, Σ_2. Suppose for some $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ we have

$$\Sigma_j \cap B_{\theta\rho} \neq \emptyset, \quad \partial\Sigma_j \subset \partial B_\rho, \quad |\partial\Sigma_j| \leq \beta\rho, \qquad \text{for } j = 1, 2.$$

Then

$$\mathcal{F}(\Sigma) \geq 8\pi - C\beta\theta$$

where C is independent of Σ, β, θ.

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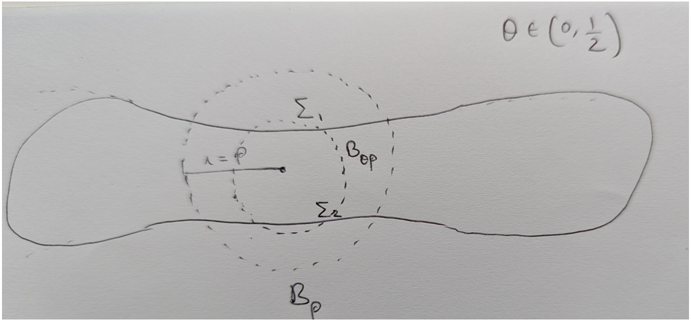

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Second fundamental form - Hessian comparison lemma

For a function with small gradient, the second fundamental form of its graph is comparable to its Hessian.

Lemma

Let $\Omega \subset \mathbb{R}^k$ *and* $u : \Omega \to \mathbb{R}^n$ *be a smooth function. Let* $T = \{(x, u(x)) : x \in \Omega\}$ *be its graph in* $\mathbb{R}^{k+n}$ *and* **A** *be the second fundamental form of* T*. Then for each* $x \in \Omega$ *we have*

$$|\mathbf{A}(x, u(x))| \leq |D^2 u(x)|.$$

Furthermore, if $|Du| << 1$*, then there is a constant* $C(k, n) > 0$ *such that*

$$|\mathbf{A}(x, u(x))| \geq C(k, n)|D^2 u(x)|.$$

Biharmonic comparison lemma

Lemma

Let L *be a plane in* $\mathbb{R}^n$ *containing* ξ, $u \in C^\infty(U)$ *where* U *is an* L*-neighborhood of* $L \cap B_\rho(\xi)$, *such that*

$$|Du| \leq 1.$$

Also, let $w \in C^\infty(L \cap \overline{B}_\rho(\xi))$ *satisfy*

$$\begin{cases} \Delta^2 w = 0 & \text{on } L \cap B_\rho(\xi) \\ w = u, \ Dw = Du & \text{on } L \cap \partial B_\rho(\xi). \end{cases}$$

Then

$$\int_{L \cap B_\rho(\xi)} |D^2 w|^2 d\mathcal{H}^2 \leq C\rho \int_\Gamma |\mathbf{A}|^2 \, d\mathcal{H}^1,$$

where $\Gamma = \mathrm{graph}(u | L \cap \partial B_\rho(\xi))$ *and* C *is a fixed constant independent of* Σ, ρ.

Roadmap

The proof of the theorem will be divided into the following steps:

1. Minimizing sequence + Monotonicity formula + measure convergence $\longrightarrow$ Hausdorff convergence.
2. Separating points with curvature concentration (called bad points) from the rest (called good points).
3. Graphical decomposition away from bad points.
4. $C^{1,\alpha} \cap W^{2,2}$ regularity at good points.
5. $C^{2,\alpha}$ regularity at good points using difference quotients.
6. Lower semicontinuity of the Willmore functional.
7. The limit surface is Willmore minimizing in its genus.
8. $C^{1,\alpha} \cap W^{2,2}$ regularity at bad points.

Measure-theoretic convergence

Let Σ_k be a minimizing sequence with

$$|\Sigma_k| = 1 \text{ and } 0 \in \Sigma_k.$$

Then by the diameter estimates, there is a uniform bound on the diameter of Σ_k. By Banach-Alaoglu theorem, there is a subsequence $\Sigma_{k'}$ and a Radon measure μ such that

$$\int_{\Sigma_{k'}} f \to \int_{\mathbb{R}^n} f d\mu$$

for each $f \in C_c(\mathbb{R}^n)$ (weak convergence of measures). Relabel the subsequence as Σ_k. Let

$$\Sigma = \operatorname{supp} \mu.$$

Hausdorff convergence

We will prove that $\Sigma_j \to \Sigma$ in Hausdorff distance. Suppose that there exists a sequence $y_j \in \Sigma_j$ such that $y_j \to y$ but $\text{dist}(y, \Sigma) = \eta > 0$. By connectedness, we can find $z_{i,k} \in \Sigma_j \cap \partial B_{(1+\frac{k}{N})\frac{\eta}{4}}(y)$. Applying the monotonicity formula at scale $\rho = \frac{\eta}{4N}$,

$$\pi \leq C\left(\rho^{-2}|\Sigma_j \cap B_\rho(z_{j,k})| + \mathcal{F}(\Sigma_j \cap B_\rho(z_{j,k}))\right).$$

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$$\pi \leq C \left(\rho^{-2} |\Sigma_j \cap B_\rho(z_{j,k})| + \mathcal{F}(\Sigma_j \cap B_\rho(z_{j,k})) \right).$$

We can take a subsequence such that $z_{j,k} \to z_k$ and $z_k \in \partial B_{(1+\frac{k}{N})\frac{\eta}{4}}(y)$. Summing over $k = 1, \ldots, N$ and using $|\Sigma_j \cap B_\rho(z_k)| \to 0$,

$$N\pi \leq C \limsup_j \mathcal{F}(\Sigma_j)$$

but N can be arbitrarily large, which is a contradiction. **(Step 1 complete)**

Separating good and bad points

A point $\xi \in \operatorname{supp}\mu$ is called a **bad point** relative to $\epsilon > 0$ if

$$\lim_{\rho\downarrow 0}\left(\liminf_{k\to\infty}\int_{\Sigma_k\cap B_\rho(\xi)}|\mathbf{A}_k|^2\right) > \epsilon^2.$$

Thanks to Gauss-Bonnet theorem, $\frac{1}{4}\int_{\Sigma_k}|\mathbf{A}_k|^2 = \mathcal{F}(\Sigma_k) - \pi(2-2g)$, the integral of $|\mathbf{A}_k|^2$ is uniformly bounded. Hence, there are finitely many bad points. In particular, if there are N bad points, then for k large enough,

$$N\epsilon^2 \leq \int_{\Sigma_k}|\mathbf{A}_k|^2 = 4\mathcal{F}(\Sigma_k) - 4\pi(2-2g).$$

Let $\xi_1,\ldots,\xi_P$, where $P = P(\epsilon)$ be the list of bad points. **(Step 2 complete)**

Graphical decomposition away from bad points

Away from the bad points, we can apply the graphical decomposition lemma (with L^2 curvature bound) to get a decomposition of Σ_k into graphs with small gradient and small holes. We can also avoid double sheeting behaviour by Simon's lemma.

Decomposition at good point

Let $\xi \in \text{spt}\mu \setminus \{\xi_1, \ldots, \xi_P\}$ be a good point. By definition, there is a $\rho(\xi, \epsilon) > 0$ such that for any $\rho \le \rho(\xi, \epsilon)$,

$$\int_{\Sigma_k \cap B_\rho(\xi)} |\mathbf{A}_k|^2 \le \epsilon^2$$

for infinitely many k.

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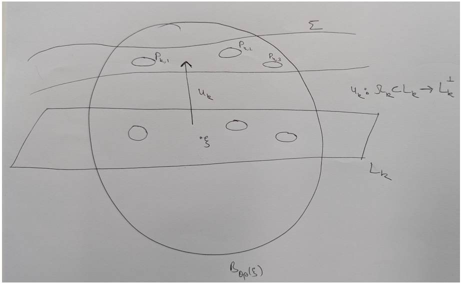
for infinitely many k.

- The graphical decomposition lemma (L^2 bound case) decomposes $\Sigma_k \cap B_\rho(\xi)$ into graphs and pimples.
- By the double sheeting lemma and $\beta_g < 8\pi$, for k large and θ small (fixed, independent of k, ϵ, ξ), only one disk $D_1^{(k)}$ meets $B_{\theta\rho}(\xi)$.

Decomposition at a good point



Decomposition at a good point

For ϵ small enough, for infinitely many k there are planes $L_k \ni \xi$ and $u_k \in C^\infty(\bar{\Omega}_k)$, $u_k : \bar{\Omega}_k \to L_k^\perp$, with

$$\rho^{-1}|u_k| + |Du_k| \leq C\epsilon^{\frac{1}{2(2n-3)}},$$

$$(\operatorname{graph} u_k \cup_j P_{k,j}) \cap B_\sigma(\xi) = D_1^{(k)} \cap B_\sigma(\xi), \qquad \sum_j \operatorname{diam} P_{k,j} \leq C\epsilon^{\frac{1}{2}}\rho,$$

where the pimples $P_{k,j}$ are disjoint closed disks away from the graphical part, and $\sigma \in (\theta\frac{\rho}{2}, \theta\rho)$ is independent of k.

Choosing a good radius σ

Let $C^k_\sigma(\xi) = \{x + y : x \in B_\sigma(\xi) \cap L_k,\, y \in L_k^\perp\}$ be the cylinder over $B_\sigma(\xi)$, and let

$$A_k = \{\sigma \in (\theta\tfrac{\rho}{2}, \theta\rho) : \partial C^k_\sigma(\xi) \cap (\cup_j P_{k,j}) = \emptyset\}.$$

Choosing ϵ small enough such that the total diameter of pimples is small ensures that $|A_k| \geq \frac{\theta\rho}{8}$.

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Choosing ϵ small enough such that the total diameter of pimples is small ensures that $|A_k| \geq \frac{\theta\rho}{8}$.
By the selection principle, there is $T \subset (\theta\frac{\rho}{2}, \theta\rho)$ with $|T| \geq \frac{\theta\rho}{8}$ such that for each $\sigma \in T$, infinitely many k satisfy

$$\partial C^k_\sigma(\xi) \cap (\cup_j P_{k,j}) = \emptyset.$$

This lets us apply the biharmonic comparison lemma to the graphical pieces $\mathrm{graph}\, u_k$.

Replacing the graphical piece

For infinitely many k, let w_k solve the biharmonic equation on $B_\sigma(\xi) \cap L_k$ with boundary data u_k, Du_k. By the biharmonic comparison lemma,

$$\int_{\mathrm{graph}\, w_k} |\tilde{\mathbf{A}}_k|^2 d\mathcal{H}^2 \sim \int_{B_\sigma(\xi)\cap L_k} |D^2 w_k|^2 d\mathcal{H}^2 \leq C\sigma \int_{\Gamma_k} |\mathbf{A}_k|^2 d\mathcal{H}^1,$$

where $\Gamma_k = \Sigma \cap \partial C_\sigma^k(\xi)$ and $\tilde{\mathbf{A}}_k$ is the second fundamental form of $\mathrm{graph}\, w_k$.

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$$\tilde{\Sigma}_k = (\Sigma_k \setminus \mathrm{graph}\, u_k) \cup \mathrm{graph}\, w_k$$

be a comparison $C^{1,1}$ surface.

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be a comparison $C^{1,1}$ surface. Minimizing also gives

$$\int_{\Sigma_k \cap B_\sigma(\xi)} |\mathbf{A}_k|^2 d\mathcal{H}^2 \le C\sigma \int_{\Gamma_k} |\mathbf{A}_k|^2 d\mathcal{H}^1 + \epsilon_k.$$

Power decay of curvature concentration at good points

The previous $\mathcal{H}^1 - \mathcal{H}^2$ inequality can be used to get a decay inequality -

$$\int_{\Sigma_k \cap B_{\frac{\rho}{2}}(\xi)} |\mathbf{A}_k|^2 d\mathcal{H}^2 \leq \gamma \int_{\Sigma_k \cap B_\rho(\xi)} |\mathbf{A}_k|^2 d\mathcal{H}^1 + \delta_k$$

where $\gamma < 1$ and $\delta_k \to 0$ as $k \to \infty$. Let

$$\psi(\xi, \rho) = \liminf_{k \to \infty} \int_{\Sigma_k \cap B_\rho(\xi)} |\mathbf{A}_k|^2 d\mathcal{H}^2,$$

then

$$\psi(\rho, \xi) \leq C \left(\frac{\rho}{\rho_0} \right)^\alpha \psi(\rho(\xi_0), \xi_0)$$

for all ξ close to a fixed good point ξ_0 and small scales $\rho \in (0, \rho_0)$. **(Step 3 complete)**

Towards $C^{1,\alpha} \cap W^{2,2}$ regularity

For all good points ξ near a fixed good point ξ_0, we have

$$\alpha_k = \alpha_k(\rho, \xi) = \int_{\Sigma_k \cap B_\rho(\xi)} |\mathbf{A}_k|^2 d\mathcal{H}^2 < \epsilon^2$$

for infinitely many k. Applying the graphical decomposition lemma with $\epsilon = \sqrt{\alpha_k}$ to Σ_k, one gets

$$\int_{B_{\theta \frac{\rho}{2}}(\xi) \cap L_k} |D\bar{u}_k - \eta_k|^2 \leq C\rho^{2+\gamma},$$

where $\bar{u}_k \in C^\infty(L_k)$ is a controlled extension of u_k on pimples.

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where $\bar{u}_k \in C^\infty(L_k)$ is a controlled extension of u_k on pimples. In measure theoretic terms, this implies

$$\mathcal{H}^2 \llcorner (\Sigma_k \cap B_\rho(\xi)) = \mathcal{H}^2 \llcorner (\mathrm{graph}\, \bar{u}_k) + \theta_k$$

with $|\theta_k| \leq C\rho^{2+\gamma}$.

Towards $C^{1,\alpha} \cap W^{2,2}$ regularity

By the compactness of the Grassmannian, we can assume that $L_k \to L$ and $\eta_k \to \eta$ and by Arzela-Ascoli theorem, $\operatorname{graph} \bar{u}_k \to \operatorname{graph} u$ with $u \in C^{0,1}$ and

$$\rho^{-1} \sup u + \sup |Du| \leq C\epsilon^{\frac{1}{2(2n-3)}}$$

and

$$\int_{B_{\theta\frac{\rho}{2}}(\xi)\cap L} |Du - \eta|^2 \leq C\rho^{2+\gamma}.$$

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This concludes the following:

The measure μ has multiplicity one tangent plane at each point $\xi \in \operatorname{supp} \mu \cap B_{\theta\frac{\rho_0}{4}}(\xi_0)$ and is locally a $C^{1,\alpha}$ surface at good points.

Towards $C^{1,\alpha} \cap W^{2,2}$ regularity

Multiplicity one allows us to bring varifold convergence into play. By Allard's compactness theorem, Σ has generalized mean curvature $\mathbf{H} \in L^2_{\text{loc}}$ at good points. The limit graph function u solves the prescribed mean curvature equation in weak sense

$$\sum_{i,j}^{2} D_i(\sqrt{g} g^{ij} D_j u) = \sqrt{g}\mathbf{H}.$$

Elliptic PDE theory now gives

$$\int_{L \cap B_\sigma(\xi)} |D^2 u|^2 \leq C\sigma^{2\alpha}.$$

Thus, Σ is a $C^{1,\alpha} \cap W^{2,2}$ surface at good points. **(Step 4 complete)**

Simon's $C^{2,\alpha}$ lemma for the 4th-order systems

Lemma

Let $u \in W^{2,2} \cap C^{1,\gamma}(\mathbb{D}; \mathbb{R}^m)$ *with* $|u| + |Du| \leq 1$ *and*

$$\int_{\mathbb{D} \cap B_\rho(\xi)} |D^2 u|^2 \leq \beta \rho^{2\gamma} \quad \forall\, \xi \in \mathbb{D},\ \rho < 1.$$

If u *weakly solves a uniformly elliptic 4th-order quasilinear system with controlled coefficients, then* $u \in W^{3,2}_{loc} \cap C^{2,\alpha}$ *for some* $\alpha > 0$*, with*

$$\int_{B_{\rho/2}(\xi)} |D^3 u|^2 \leq C \rho^{2\alpha}.$$

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The above lemma implies $C^{2,\alpha} \cap W^{2,2}$ regularity at good points which can be bootstrapped to C^∞ regularity and analyticity. **(Step 5 complete)**

Lower semicontinuity of the Willmore functional

By Allard's compactness theorem, the first variation converges so for any fixed $\Phi : \mathbb{R}^n \to \mathbb{R}^n$,

$$\lim_k \int_{\Sigma_k} \Phi \cdot \mathbf{H}_k = \int_{\Sigma} \Phi \cdot \mathbf{H}.$$

Extend $\mathbf{H}$ to $\mathbb{R}^n \setminus \{\xi_1, \ldots, \xi_P\}$ and take test function $\Phi = \zeta \mathbf{H}$ with $\zeta \in C_c^\infty(\mathbb{R}^n \setminus \{\xi_1, \ldots, \xi_P\})$. By Cauchy-Schwarz and controlling the cutoff function one can show that

$$\int_{\Sigma \setminus \cup B_\rho(\xi_i)} |\mathbf{H}|^2 \leq \liminf_{k \to \infty} \int_{\Sigma_k \setminus \cup B_\rho(\xi_i)} |\mathbf{H}_k|^2$$

for small ρ. Pushing $\rho \to 0$ gives lower semicontinuity.

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Repeating the same for curvature varifolds gives

$$\int_{\Sigma \setminus \cup B_\rho(\xi_i)} |\mathbf{A}|^2 \leq \liminf_{k \to \infty} \int_{\Sigma_k \setminus \cup B_\rho(\xi_i)} |\mathbf{A}_k|^2.$$

(Step 6 complete)

Next steps

1. The limit surface is Willmore minimizing in its genus.
2. $C^{1,\alpha} \cap W^{2,2}$ regularity at bad points.
3. Conformally change a Willmore minimizing sequence of tori so that there is no genus loss.

What has been proved so far

A genus g minimising sequence Σ_k, $|\Sigma_k| = 1$, $0 \in \Sigma_k$, with limit $\Sigma = \operatorname{spt}\mu$ and finitely many bad points $\xi_1, \ldots, \xi_P$.

- Hausdorff and measure-theoretic convergence $\Sigma_k \to \Sigma$.
- Σ is real analytic *away from* $\xi_1, \ldots, \xi_P$.
- Lower semicontinuity:

$$\int_{\Sigma \cap U} |\mathbf{H}|^2 \leq \lim_{\rho \downarrow 0} \liminf_{k} \int_{\Sigma_k \cap U \setminus \cup B_\rho(\xi_i)} |\mathbf{H}_k|^2,$$

  and the same with $\mathbf{A}$ in place of $\mathbf{H}$.

Consequences of monotonicity at ξ_i

The varifold convergence has two important consequences:

1. **Positive density.** From $\pi \leq C(\rho^{-2}|\Sigma \cap B_\rho(\xi_i)| + \mathcal{F}(\Sigma \cap B_\rho(\xi_i)))$ and $\mathcal{F}(\Sigma \cap B_\rho(\xi_i)) \to 0$:
$$|\Sigma \cap B_\rho(\xi_i)| \geq C\rho^2.$$
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2. **Approximately conical.** With $X = x - \xi_i$, the monotonicity identity has bounded right-hand side, so
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Here $|X^\perp|/|X|$ is the sine of the angle between the position vector and $T_x\Sigma$; it vanishes iff Σ is a cone with vertex ξ_i. Since $|X| \approx \sigma$ on an annulus,
$$\frac{|X^\perp|}{|X|} \leq \epsilon \quad \text{on } B_\sigma(\xi_i) \setminus B_{\frac{\sigma}{2}}(\xi_i), \quad \text{off a set of measure } \leq C\epsilon\sigma^2.$$

L^2 curvature bound at a bad point

From (curvature) varifold convergence, we have $\int_\Sigma |\mathbf{A}|^2 < \infty$, so for small σ

$$\int_{\Sigma \cap B_\sigma(\xi_i)} |\mathbf{A}|^2 \leq \frac{\epsilon^2}{4}, \qquad i = 1, \ldots, P.$$

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Compare with the definition of a bad point:

$$\lim_{\rho \downarrow 0} \liminf_k \int_{\Sigma_k \cap B_\rho(\xi_i)} |\mathbf{A}_k|^2 > \epsilon^2.$$

This suggests that the loss of genus is happening at the bad points.

One sheet over an annulus

There is an annulus $\mathcal{A}_i = \{x \in L_i : (\frac{3}{4} - \frac{\theta}{2})\sigma < |x - \xi_i| < (\frac{3}{4} + \frac{\theta}{2})\sigma\}$ in the plane L_i, and its **solid slab**

$$\tilde{\mathcal{A}}_i = \{x + z : x \in \mathcal{A}_i,\ z \in L_i^\perp,\ |z| \le \theta\sigma/2\},$$

over which Σ is a *single* C^1 graph u_i plus pimples:

$$\Sigma \cap \tilde{\mathcal{A}}_i = \Big(\operatorname{graph} u_i \cup \bigcup_k P_{i,k}\Big) \cap \tilde{\mathcal{A}}_i, \qquad \sum_k \operatorname{diam} P_{i,k} \le C\epsilon^{\frac{1}{2(2n-3)}}\sigma.$$

Over the *annulus* — which excludes a disc about ξ_i — the limit is exactly one sheet.

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Over the *annulus* — which excludes a disc about ξ_i — the limit is exactly one sheet.

Why only one sheet? Two sheets through ξ_i would put us in the double-sheet situation, forcing

$$\mathcal{F}(\Sigma) \ge 8\pi - C\beta\theta > \beta_g,$$

a contradiction for θ small.

Convergence of curvature

Using the fact that Σ_k is a minimizing sequence, together with graphicality over the annulus around the bad points, Simon proves upper semicontinuity inequality to obtain equality:

convergence with curvature density

$$|\mathbf{H}_k|^2\mathcal{H}^2 \llcorner \Sigma_k \to |\mathbf{H}|^2\mathcal{H}^2 \llcorner \Sigma, \qquad |\mathbf{A}_k|^2\mathcal{H}^2 \llcorner \Sigma_k \to |\mathbf{A}|^2\mathcal{H}^2 \llcorner \Sigma$$

on $\mathbb{R}^n \setminus \{\xi_1, \ldots, \xi_P\}$.

The only possible loss of energy is at the bad points!

Σ is in fact Willmore minimizing in its genus

Let $\tilde{\Sigma}$ be any competitor of the same genus. Near ξ_i, $\Sigma \cap \partial B_\sigma(\xi_i)$ is a single Jordan curve, C^1-close to a plane.
Replace $\tilde{\Sigma} \cap B_{\sigma_k}(y_i)$ by a deformation of $\Sigma_k \cap B_{\sigma_k}(\xi_i)$:

$$\tilde{\Sigma}_k = \Big(\tilde{\Sigma} \setminus \bigcup_i B_{\sigma_k}(y_i)\Big) \cup \Big(\bigcup_i \big(\Sigma_k \cap B_{\sigma_k}(\xi_i)\big)^*\Big).$$

Minimality of Σ_k against $\tilde{\Sigma}_k$ gives

$$\begin{aligned}\mathcal{F}\Big(\Sigma_k \cap \bigcup B_{\sigma_k}(\xi_i)\Big) + \mathcal{F}\Big(\Sigma_k \setminus \bigcup B_{\sigma_k}(\xi_i)\Big) &= \mathcal{F}(\Sigma_k) \le \mathcal{F}(\tilde{\Sigma}_k) + \epsilon_k \\ &\le \mathcal{F}\Big(\tilde{\Sigma} \setminus \bigcup B_{\sigma_k}(y_i)\Big) + \mathcal{F}\Big(\Sigma_k \cap \bigcup B_{\sigma_k}(\xi_i)\Big) + \tilde{\epsilon}_k.\end{aligned}$$

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The bad-ball caps — carrying *all* the concentrated energy — appear on both sides and **cancel**. Since $\mathcal{F}(\Sigma_k \setminus \bigcup B_{\sigma_k}(\xi_i)) \to \mathcal{F}(\Sigma)$:

$$\mathcal{F}(\Sigma) \le \mathcal{F}(\tilde{\Sigma}).$$

(Step 7 complete)

Σ is smooth even at bad points

Another iteration of biharmonic comparison lemma at ξ_i gives a decay inequality:

$$\int_{\Sigma \cap B_\rho(\xi_i)} |\mathbf{A}|^2 \leq C\rho^{2\gamma}.$$

Repeating the regularity estimates at good points gives $C^{1,\alpha} \cap W^{2,2}$ regularity at bad points and then $C^{2,\alpha}$ regularity at bad points. Bootstrapping gives real analyticity at bad points. **(Step 8 complete)**

Preventing genus loss

Theorem

If Σ_k is a minimizing sequence of tori such that

$$\Sigma_k \longrightarrow \Sigma, \text{ a sphere.}$$

Then there is a new minimizing sequence $\tilde{\Sigma}_k \subset B_1(0)$ obtained after conformal transformation such that

$$\tilde{\Sigma}_k \longrightarrow \tilde{\Sigma}, \text{ a torus.}$$

Inversion trick

For an arbitrary surface $\Sigma \subset \mathbb{R}^n$. For $y \notin \Sigma$, let $d_\Sigma(y) = \mathrm{dist}(y, \Sigma)$ and

$$S_\epsilon(y) = \{q \in \Sigma : |y - q| < (1 + \epsilon^3) d_\Sigma(y)\}.$$

If $d_\Sigma(y) < \frac{1}{4} \operatorname{diam}(\Sigma)$, we consider the inversion

$$\begin{aligned} x &\mapsto d_\Sigma(y) \frac{x - y}{|x - y|^2} \\ \Sigma &\to \tilde{\Sigma} \end{aligned}$$

Dichotomy

Non-spherical condition

$\mathrm{diam}(\tilde{\Sigma}) \geq \frac{1}{4}$, Hausdorff-dist$(\tilde{\Sigma}, S) \geq \frac{1}{64}$ for *every* round sphere $S \subset \bar{B}_1(0)$.

Here's the dichotomy: either the inverted surface $\tilde{\Sigma}$ is non-spherical or

$$\mathrm{diam}(S_\epsilon(y)) < \epsilon\, d_\Sigma(y) \quad \forall\, y \ \text{ with } d_\Sigma(y) < \tfrac{1}{4}\,\mathrm{diam}(\Sigma).$$

Handle estimates

For a torus $\Sigma \subset \mathbb{R}^n$, there exists a smooth map $\Gamma_0 : \mathbb{S}^{n-2} \to \mathbb{R}^n \setminus \Sigma$ which is not null-homotopic. Define

$$\delta_0 = \sup_{\Gamma \in [\Gamma_0]} \operatorname{dist}(\Gamma, \Sigma),$$

a kind of width of the handle in $\mathbb{R}^n \setminus \Sigma$ (distance between the two sets).

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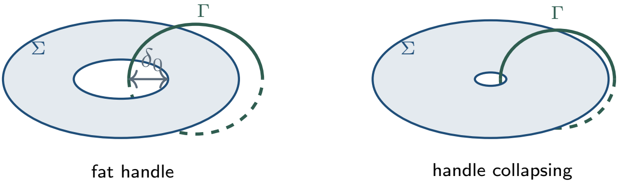

Ruling out the alternative

Suppose the alternative holds and $\delta_0 < \frac{1}{16}\,\mathrm{diam}(\Sigma)$. Simon constructs a homotopy pushing Γ *further* from Σ, producing a competitor of clearance $> \delta_0$ — contradicting maximality of δ_0.

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So the alternative fails, and a good inversion exists:

$$\delta_0 < \frac{1}{16}\operatorname{diam}(\Sigma) \implies \tilde{\Sigma} \text{ is non-spherical.}$$

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So the alternative fails, and a good inversion exists:

$$\delta_0 < \frac{1}{16}\operatorname{diam}(\Sigma) \Longrightarrow \tilde{\Sigma} \text{ is non-spherical.}$$

A minimizing sequence of tori Σ_k converging to a sphere has two sided diameter bounds so it *must* have small handle width eventually!

Summary

minimising sequence $\Sigma_k \;\Rightarrow\;$ analytic limit Σ, genus $g_0 \leq g$

↓

curvature concentrates at bad points $=$ handles pinching off $=$ genus loss

↓

conformal invariance: inversions and dilations preserve $\mathcal{F}$

↓

genus $= 1 \Rightarrow$ non-null-homotopic $\mathbb{S}^{n-2} \Rightarrow$ small width $\delta_0 \Rightarrow$ a good inversion exists

↓

new sequence in $\bar{B}_1(0)$, quantitatively far from every round sphere

↓

limit is an analytic Willmore-minimising torus

Thanks for your attention!

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- M. Bauer, E. Kuwert, *Existence of minimizing surfaces of prescribed genus*, Int. Math. Res. Not. 2003, no. 10, 553–576.